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On Study C^h-Trirecurrent Finsler Space

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ABSTRACT

The concept of C^h- recurrent Finsler space has been studied by M.Matsumoto [6] . H.Izumi ([4],[5]) gave the concept of *P- spaces which was the generalization of C^h-recurrent spaces and P2-like spaces of M.Matsumoto ([6],[7]). R.Verma [15] discussed C^h-birecurrent spaces where these spaces are generalization of C^h-recurrent spaces of M.Matsumoto [6] .Besides the correlation of C^h- birecurrent spaces which C^h- recurrent space , some special C^h- birecurrent spaces has been discussed . The result concerning h- isotropic C^h- recurrent space due to M.Matsumoto [6] has been extended to C^h- birecurrent spaces by P.N. Pandey and R.Verma [15]. C.K.Mishra and G.Lodhi [9] studied the properties of C^h-recurrent and C^v- recurrent for torsion tensor field of the second order in Finsler spaces .

The purpose of the present paper is to study the properties of C^h- trirecurrent torsion tensor field and the recurrence covariant vector field of the third order in Finsler spaces .

Keywords: h- Trirecurrent Tensor , C^h-Trirecurrent Finsler Space , C^h-Trirecurrent Affinely Connected Space and P*- C^h- Trirecurrent Space.

INTRODUCTION

Let us consider an n-dimensional Finsler space F_n equipped with a metric function F(x^i, y^i) satisfying the requestic conditions of a Finslerian metric[10],the corresponding symmetric metric tensor g_{ij} ** and Cartan's connections parameters .

The relations between the metric function F and the corresponding metric tensor g_{ij} are given by

(1.1) $$g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2 \quad {}^{***}$$.

Corresponding to each contravariant vector y^i , there is a covariant y_i , such that

(1.2) $y_i = g_{ij} y^j$.

The (h) hv –torsion tensor C_{ijk} defined by M.Matsumoto [6]

(1.3) $C_{ijk} := \frac{1}{2}\dot{\partial}_i g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k F^2$,

it is positively homogenous of degree -1 in y^i and symmetric in all its indices .
The (v) hv- torsion tensor C^i_{jk} which is the associate tensor of C_{ijk} and is defined by

(1.4) $C^i_{jk} := g^{ip} C_{jpk}$.

For an arbitrary vector field X^i , É. Cartan deduced ([1] , [2])

(1.5) $X^i_{|k} := \partial_k X^i - \left(\dot{\partial}_r X^i\right) G^r_k + X^r \Gamma^{*i}_{rk}$,

where the functions Γ^{*i}_{rk} and G^r_k are defined by

(1.6) a) $\Gamma^{*i}_{rk} := \Gamma^i_{rk}$ -$C^i_{mr}\, \Gamma^m_{sk} y^s$

and

b) $G^r_k = \Gamma^{*r}_{sk}\, y^s$.

The functions Γ^{*i}_{rk} defined by (1.6a) are called *Cartan's connection parameters*. These are symmetric in its lower indices are positively homogenous of degree zero in y^i .
The equation (1.5) gives a process of covariant differentiation known as *h-covariant differentiation (Cartan's second kind covariant differentiation)* .M.Matsumoto ([6] ,[7]) calls this derivative as " *h-covariant derivative*" .

The associate tensor g^{ij} of the metric tensor g_{ij} is covariant constant with respect to the above process , i.e.

(1.7) $g^{ij}_{|k} = 0$.

The h-covariant derivative of the vector y^i vanishes identically , i.e.

(1.8) $y^i_{|k} = 0$.

The commutation formula for Cartan's covariant differentiation of an arbitrary vector field X^i expressed as follows:

(1.9) $X^i_{|j|k} - X^i_{|k|j} = R^i_{hjk}\, X^h - H^h_{jk}\, X^i_{|h}$.

The h-curvature tensor R^i_{jkh} (which is the third of Cartan's curvature tensors) is defined by

(1.10) $R^i_{jkh} := \partial_h \Gamma^{*i}_{jk} + \left(\dot{\partial}_l\, \Gamma^{*i}_{jk}\right) \Gamma^{*l}_{sh} y^s + C^i_{jm}\left(\partial_k \Gamma^{*m}_{sh} y^s - \Gamma^{*m}_{kl} \Gamma^l_{sh} y^s\right) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - {}^k/_h$.

The (hv)- curvature tensor P^i_{jkh} (which is the second of Cartan curvature tensor) is defined by

(1.11) $P^i_{jkh} := C^i_{kh|j} - g^{ir} C_{jkh|r} + C^r_{jk} P^i_{rh} - P^r_{jh} C^i_{rk}$.

This tensor satisfies tensor

(1.12) $P^i_{jkh} y^j = P^i_{kh} = C^i_{kh|r}\, y^r$.

The tensor P^i_{kh} is called *v (hv) – torsion tensor.*

1. C^h –Trirecurrent Finsler Space

M.Matsumoto [6]defined an –recurrent Finsler space by the condition

(2.1) a) $C_{ijk|l} = \lambda_l C_{ijk}$, $C_{ijk} \neq 0$

or equivalent to the condition [9]

(2.1) b) $C^i_{jk|l} = \lambda_l C^i_{jk}$, $C^i_{jk} \neq 0$,

where λ_l is non-zero covariant vector field .

R.Verma [15] defined an C^h –birecurrent Finsler space by the condition

(2.2) a) $C_{ijk|l|m} = a_{lm} C_{ijk}$, $C_{ijk} \neq 0$
or equivalent to the condition [10]
(2.2) b) $C^i_{jk|l|m} = a_{lm} C^i_{jk}$, $C^i_{jk} \neq 0$,
where $a_{lm} = \lambda_{l|m} + \lambda_l \lambda_m$ is a recurrence covariant tensor field of second order.
Taking h- covariant differentiation of (2.1a) with respect to x^m , we get
(A) $C_{ijk|l|m} = \lambda_{l|m} C_{ijk} + \lambda_l \lambda_m C_{ijk}$.
Again taking h-covariant differentiation of (A) with respect to x^n , we get
(2.3) $C_{ijk|l|m|n} = a_{lmn} C_{ijk}$, $C_{ijk} \neq 0$,
where
(B) $a_{lmn} = \lambda_{l|m|n} + \lambda_{l|m} \lambda_n + \lambda_{l|n} \lambda_m + \lambda_l \lambda_{m|n} + \lambda_l \lambda_m \lambda_n$.

Definition 2.1. The space in which the (h) hv- torsion tensor C_{ijk} satisfies the condition (2.3) , where a_{lmn} is recurrence covariant tensor field of third order defined by the equation (B) , the space and the tensor satisfying the condition (2.3) will be called C^h – *trirecurrent* and *h-trirecurrent tensor* respectively , we shall denote such space and tensor briefly by C^h–*TR*-F_n and *h-TR* respectively.

If we assume the condition (2.3) which is the characterizing equation of C^h–TR-F_n , where a_{lmn} is the recurrence covariant tensor field of third order , it does not imply the condition (2.1a) in general.
Therefore the condition (2.3) is more general than the condition (2.1a) .In this case the recurrence covariant tensor field a_{lmn} of third order need not to be of the form (B) .
Thus , we conclude

Theorem 2.1. Every C^h–recurrent Finsler space (for which the recurrence vector field satisfies (B) is not zero) , is C^h–TR-F_n .

Corollary 2.1. In C^h- TR-F_n , the (v) hv-torsion tensor is h-TR .
Proof
Let us consider C^h- TR-F_n characterized by (2.3) .
Transvecting (2.3) by g^{qj} and using (1.7) and (1.4) , we get
(2.4) $C^q_{ik|l|m|n} = a_{lmn} C^q_{ik}$, $C^q_{ik} \neq 0$.
Now , transvecting (2.4) by y^l and using (1.8) and (1.12) , we get
(2.5) $P^q_{ik|l|m} = a_{lmn} y^l C^q_{ik}$,
Also , let us consider a C^h- TR-F_n characterized by (2.3) which is also a P* –Finsler space.
For such space we have the condition (2.5) and the equation
(2.6) $P^q_{ik} = \phi C^q_{ik}$,
where ϕ is non-zero scalar.
Taking h-covariant differentiation of (2.6) with respect to x^m , we get
(2.7) $P^i_{ik|m} = \phi_{|m} C^q_{ik} + \phi C^q_{ik|m}$.
Transvecting (2.7) by y^m and using (2.10) , we get
(2.8) $P^i_{ik|m} y^m = \phi_{|m}\, y^m C^q_{ik} + \phi P^q_{ik}$.
In view of (2.6) , the equation (2.8) can be written as
(2.9) $P^i_{ik|m} y^m = \phi_{|m}\, y^m C^q_{ik} + \phi\phi\, C^q_{ik}$.
Taking h-covariant differentiation of (2.9) with respect to x^n and using (1.8) , we get
(2.10) $P^q_{ik|m|n}\, y^m = \phi_{|m|n}\, y^m C^q_{ik} + \phi_{|m}\, y^m C^q_{ik|n} + 2\phi\phi_{|n}\, C^q_{ik} + \phi\phi\, C^q_{ik|m}$.

In view of (2.5) and (2.10) , we get

$$a_{lmn} y^l y^m C^q_{ik} = \phi_{|m|n} y^m C^q_{ik} + \phi_{|m} y^m C^q_{ik|n} + 2\phi\phi_{|n} C^q_{ik} + \phi^2 C^q_{ik|m}$$

or

$$C^q_{ik|n} = \left(\frac{a_{lmn} y^l y^m - \phi_{|m|n} y^m - 2\phi\phi_{|n}}{\phi_{|m} y^m + \phi^2}\right) C^q_{ik}$$

which shows that the space is C^h −recurrent provided

$$\left(\frac{a_{lmn} y^l y^m - \phi_{|m|n} y^m - 2\phi\phi_{|n}}{\phi_{|m} y^m + \phi^2}\right) = 0 .$$

Thus , we conclude

Theorem 2.2. The C^h–TR-F_n *is* C^h–recurrent if it is a P* –Finsler space and $\phi_{|n} \neq 0$, ϕ being defined in (2.6) .

Commutating (2.4) with respect to the indices m and n and using commutation formula (1.9) , we get

(2.11) $$C^h_{ik} R^q_{hmn||l} - C^q_{hk} R^h_{imn||l} - C^q_{ih} R^h_{kmn||l} - C^q_{ik\, l} H^h_{mn||l} - C^h_{ik||l} R^h_{hmn} - C^q_{hk||l} R^h_{imn} - C^q_{ih||l} R^h_{kmn} - C^q_{ik\, h||l} H^h_{mn} = (a_{lmn} - a_{lnm}) C^q_{ik} .$$

Note 2.1. An affinely connrcted space is characterized by any one of the following equivalent conditions

a) $G^i_{jkh} = 0$ and b) $C_{ijk|h} = 0$.

Thus , we may conclude

Theorem 2.3. If the C^h–TR-F_n is affinely connected space , the recurrence covariant tensor field of third order a_{lmn} is symmetric in its last two indices .

Contracting the indices q and i in (2.11) and putting C_k for C^q_{qk} , we get

(2.12) $$(a_{lmn} - a_{lnm})\, C_k = -C_{h||l} R^h_{hmn} - C_{k\, h||l} H^h_{mn} - C_h R^h_{kmn||l} - C_{k\, h} H^h_{mn||l} .$$

Due to the skew –symmetric of R_{hkmn} in its last two indices , we have

(2.13a) $$C_h R^h_{kmn||l} C^k = R_{hkmn||l} C^h C^k = 0$$

and

(2.13b) $$C_{h||l} R^h_{kmn} C^k = R_{hkmn} C^h_{||l} C^k = 0 ,$$

where $C^k = g^{ik} C_i$.

Transvecting (2.12) by C^k and using (2.13a) and (2.13b) , we get

(2.14) $$(a_{lmn} - a_{lnm})\, C_k C^k = -C_{k\, h||l} C^k H^h_{mn} - C_{k|h} C^k H^h_{mn||l}$$

which can be written as

(2.15) $$(a_{lmn} - a_{lnm})\, C_k C^k = -C_{k\, h||l} C^k R^h_{rmn} y^r - C_{k|h} C^k R^h_{rmn||l} y^r .$$

Transvecting (2.15) by C_r and using (2.13) , we get

$$(a_{lmn} - a_{lnm})\, C_k C^k C_r = 0 .$$

This implies at least one of the following :

(2.16) a) $a_{lmn} - a_{lnm} = 0$ and b) $C_k C^k C_r = 0$.

The condition (2.16a) , implies that the recurrence covariant tensor field a_{lmn} of third order is symmetric in its last two indices .

The condition (2.16b) , implies $C_r = 0$ which in view of Deicke's theorem [4] implies that the space is Riemannian .

Thus , we conclude

Theoram 2.4. A C^h–TR-F_n either its recurrence covariant tensor field of third order is symmetric in its last two indices or Riemannian space .

Suppose that there exists a non-null covariant vector field λ_l such that

(2.17) a) $H^i_{rmn\|l} + H^i_{rhm\|l} + H^i_{rnh\|l} = 0$

and

b) $\lambda_h H^i_{rmn} + \lambda_n H^i_{rhm} + \lambda_m H^i_{rnh} = 0$.

Transvecting (2.15) by λ_q , we have

(2.18) $b_{lmn}\lambda_q C_k C^k = -C_{k\,h\|l}\,\lambda_q C^k H^h_{rmn} y^r - C_{k\,h}\lambda_q C^k H^h_{rmn\|l} y^r$,

where $b_{lmn} = a_{lmn} - a_{lnm}$.

Taking skew-symmetric part of (2.18) with respect to the indices m, n and q , we get

$$(b_{lmn}\lambda_q + b_{lqm}\lambda_n + b_{lnq}\lambda_m)\, C_k C^k = -C_{kh} C^k y^r (\lambda_q H^h_{rmn} + \lambda_n H^h_{rqm} + \lambda_m H^h_{rnq})$$
$$-C_{k|h\|l} C^k y^r (\lambda_q H^h_{rmn} + \lambda_n H^h_{rqm} + \lambda_m H^h_{rnq}) .$$

In view of (2.17) , the above equation implies

(2.19) $(b_{lmn}\lambda_q + b_{lqm}\lambda_n + b_{lnq}\lambda_m)\, C_k C^k = 0$.

This implies at least one of the following :

(2.20) a) $b_{lmn}\lambda_q + b_{lqm}\lambda_n + b_{lnq}\lambda_m = 0$

and

b) $C_k C^k = 0$.

The condition (2.20b) implies $C_k = 0$ which in view of Deicke's theorem[4] implies that the space is Riemannian.

That is , if a C^h–TR-F_n admits the identity (2.17) , the space is either admits (2.20a) or Reimannian space .

Thus , we conclude

Theorem 2.5. If a C^h–TR-F_n admits the identity (2.17) , the space either admits (2.20a) or Reimannian space .

Since a R^h–recurrent Finsler space [15] , a K^h–recurrent Finsler space [11] and a H–recurrent Finsler space [13] admit the identity (2.17) .

Similarly, we may conclude

Corollary 2.2. *A* C^h–TR-F_n is either admits (2.20a) or Reimannian space provided if satisfies one of the following :

(1) It is a R^h–recurrent Finsler space ,

(2) It is a K^h–recurrent Finsler space ,

(3) It is a H–recurrent Finsler space .

If the deviation tensor H^i_h of C^h–TR-F_n vanishes identically . In view of $H^i_{kh} = \frac{1}{3}(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)$, the equation (2.14) reduces to $(a_{lmn} - a_{lnm})\, C_k C^k = 0$. This implies that the space either its recurrence covariant tensor field of third order is symmetric in its last two or Riemannian space .

In the later case , the equation (2.12) reduces

(2.21) $C_{h\|l} R^h_{kmn} = C_h R^h_{kmn\|l}$.

Thus , we conclude

Theoram 2.6. $A\ C^h$–TR-F_n with vanishing deviation tensor if either its recurrence tensor field of third order is symmetric in its last two indices or Reimannian space , then the curvature tensor R^i_{jkh} satisfies (2.21).

Taking h- covariant differentiation of (2.2a) with respect to x^n , we get

(2.24) $C_{ijk|l|m|n} = a_{lm|n}C_{ijk} + a_{lm}C_{ijk|n}$, $C_{ijk} \neq 0$.

If the (h) hv- torsion tensor C_{ijk} is h-TR , the equation (2.24) can be written as

(2.25) $C_{ijk|l|m|n} = b_{lmn}C_{ijk}$, $C_{ijk} \neq 0$,

here

(C) $b_{lmn} = a_{lm|n} + a_{lm}\lambda_n$.

If we assume the condition (2.25) is characterizing equation of C^h–TR-F_n , where $b_{lmn}$is the recurrence covariant tensor field of third order , it does not imply the condition (2.2a) in general . Therefore the condition (2.25) is more general than the condition (2.2a) . In this case the recurrence covariant field b_{lmn} of third order need not to be of the form (C).

Thus, we conclude

Theorem 2.6. If the (h) hv- torsion C_{ijk} is h-TR , then every C^h–TR-F_n (for which the reucerrence vector field satisfies the equation (C) is not zero). is C^h–TR-F_n.

Corollary 2.3. In C^h–TR-F_n , the (v) hv- torsion tensor C^i_{jk} is h-TR provided C^i_{jk} is h-recurrent .

Proof

Let us consider C^h–TR-F_n characterized by (2.3) .

Transvecting (2.3) by g^{qj} and using (1.7) and (1.4) , we get

(2.26) $C^q_{ik|l|m|n} = b_{lmn}C^q_{ik}$, $C^q_{ik} \neq 0$.

Let us transvecting (2.26) by y^l and using (1.8) and (1.12) , we get

(2.27) $P^q_{ik|l|m|n} = b_{lmn}\, y^l\, C^q_{ik}$.

Let us consider a C^h–TR-F_n characterized by (2.3) which is also a P* –Finsler space. For such space we have the condition (2.27) and the equation (2.6) .

In view of (2.6) , the equation (2.8) can be written as

$$P^q_{ik|m}\ \phi\ y^m = (\phi_{|m}\, y^m + \phi^2)\ P^q_{ik} \quad .$$

Note 2.2. P* –Finsler space is characterized by the condition ([4], [5])

$$P^i_{kh} = C^i_{kh|j}y^i = \phi C^i_{kh} \quad , \phi \neq 0 .$$

Thus , we conclude

Theorem 2.8. If the C^h–TR-F_n is P* –Finsler space , the h- covariant derivative of the v(hv) –torsion tensor P^q_{ik} is proportional to the tensor P^q_{ik} for which the recurrence $\phi_{|m}\, y^m + \phi^2 \neq 0$.

In view of (2.6) , the equation (2.5) can be written as

(2.28) a) $P^q_{ik|m|n} = \frac{1}{\phi}\, a_{lmn}y^l P^q_{ik}$

or

b) $\phi P^q_{ik|m|n} = a_{lmn}y^l P^q_{ik}$.

Thus, we conclude

Theorem 2.9. If the C^h–TR-F_n is P* –Finsler space , the v(hv) –torsion tensor P_{ik}^q is birecurrent for which the recurrence covariant tensor field of second order $a_{lmn}\frac{y^l}{\phi}$ is not zero.

or

Theorem.2.10. If the C^h–TR-F_n is P* –Finsler space , the second h- covariant derivative of the v(hv) –torsion tensor P_{ik}^q is proportional to the second directional derivative of the tensor P_{ik}^q in the directional of y^n and y^m .

REFERENCES

[1] Cartan, É. (1933) Sur les espaces de Finsler, C.R .Acad. Sci. paris .196 . 582-586.

[2] Cartan, É. (1934) 2nd edit (1971) . *Les espaces de Finsler* . *Actualites* . *Paris* .

[3] Deicke, A. (1951) Über die Finsler Räume mit $A_i = 0$. Arch.Math .4: 45-51.

[4] Izumi, H. (1976) On *P - *Finsler spaces I. Defence Academy of Japan*.16(4): 133- 138.

[5] Izumi, H. n *P - *Finsler spaces II . Defence Academy of Japan*. 17(1) : 1- 9.

[6] Matsumoto, M. (1971) On h-isotropic and C^h-recurrent Finsler spaces . J. Math. Kyoto Univ . 11 : 1- 9.

[7] Matsumoto, M. (1972) On C-reducible Finsler spaces . Tensor N.S.24: 29-37.

[8] Matsumoto, M. and Namuta ,S. (1980) On C-reducible Finsler spaces with constant coefficients and C2-like Finsler spaces . Tensor N.S.34: 218-222.

[9] Mishra,C.K. and Lodhi G. (2008) On C^h −recurrent and C^v −Recurrent Finsler Spaces of second order .Int. J.Contemp. Math.Sciences. Vol. 3 . No. 17: 801-810.

[10] Mishra,R.S. and Pande,H.D. (1968) Recurrent Finsler spaces.J.Int. Math. Soc.32: 17-22.

[11] Misra, R.B. (1973) On a recurrent Finsler space. Rev.Roum. Math .Pure.Appl.18: 701-712.

[12] Pandey, P.N. and Verma, R. (1997) C^h- birecurrent Finsler spaces .second conference of the International Academy Physical Sciences.

[13] Qasem,F.Y.A. And . Muhib , A.A.A. (2008). On R^h- Trirecurrent Finsler spaces. Sci.J .Fac.Edu. Vol.(I) No. (5) : 59-75.

[14] Rund, H. (1959). *The Differential geometry of Finsler space.* Springer-Verlag .Berlin. Göttingen –Heidelberg 383 pp.

[15] Verma,R. (1991). *Some transformations in Finsler spaces* .D.Phil.Thesis .University of Allahabad .Allahabad .India.111p.

ثلاثي المعاودة C^h — دراسة حول فضاء فنسلر

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ملخص

في هذه الورقة قدمنا فضاء فنسلر F_n الذي يحقق فيه $(h)hv$– الموتر الالتوائي C_{ijk} الحالة التالية: $C_{ijk|l|m|n} = a_{lmn}C_{ijk}$, $C_{ijk} \neq 0$, على x^l, x^m, x^n من الرتبة الثالثة بالنسبة إلى المؤثر التفاضلي المتحد الاختلاف h–هو $|l|m|n$ حيث ثلاثي C^h–هو حقل متجهي متعدد الاختلاف غير صفري من الرتبة الثالثة. أسميناه بفضاء a_{lmn} التعاقب ، موتر ثلاثي المعاودة. h – المعاودة وأطلقنا على الموتر الذي يحقق خاصية ثلاثي المعاودة بـــ ثلاثي المعاودة وذلك من خلال دراسة : خواصه في أنواع معينة C^h– الغرض من هذه الورقة هو تطوير فضاء من الرتبة a_{lmn} وكذلك حقل الموتر المتجهي C^i_{jk} الموتر ألالتوائي $(v)hv$– في فضاء فنسلر وأيضا سلوك الثالثة. ثلاثي المعاودة C^h–ثلاثي المعاودة، فضاء C^h– موتر ثلاثي المعاودة، h – فضاء فنسلر.

كلمات مفتاحية : ثلاثي المعاودة. P^* – C^h–، فضاء الـ Affinely Connected