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ON GENERALIZED N−PREOPEN SETS

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ABSTRACT

The class of N−preopen sets was introduced in topological spaces. The purpose of this paper is to introduce and study the notion for the new class of N−preopen sets which is finer than the class of generalized preopen sets and the class of generalized open sets. Furthermore, we study the basic topological properties and introduce the notion of generalized N−precontinuous functions.
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1. INTRODUCTION

Let A be a subset of a topological space (X, τ). The closure and the interior of A will be denoted by Cl(A) and Int(A), respectively. A is called preopen set [3] if A $\subseteq$ Int(Cl(A)). The complement of preopen set is called preclosedset. Recall [3] that A is preclosed set if and only if Cl(Int(A)) $\subseteq$ A. The p−closure set of A is defined as the intersection of all preclosed subsets of X containing A and is denoted by Clp(A). The p−interior set of A is defined as the union of all preopen subsets of X contained in A and is denoted by Intp(A).

A subset A of topological space (X, τ) is called a N−preopen set [7] if for each x $\in$ A, there exists a preopen set Ux containing x such that Ux − A is a finite set. The complement of N−preopen set is called N−preclosed set. The N−closure set of A is defined as the in-tersection of all N−preclosed subsets of X containing A and is denoted by Cln(A). The N−interior set of A is defined as the union of all N−preopen subsets of X contained in A and is denoted by Intn(A).

In 1970, Levine [2] introduced the notion of generalized closed sets. A subset A of a topological space (X, τ) is called generalized closed (simply g−closed) set if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of X. The complement of g−closed set is called generalized open (simply g−open) set. In [4], they introduced the notion of generalized preclosed sets. A subset A of a topological space (X, τ) is called generalized preclosed (simply g−preclosed) set if $Clp(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of (X, τ). The complement of g−preclosed set is called generalized preopen (simply g−preopen) set. This paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we give the concept of generalized N−preopen sets by utilizing the N−closure operator and we study its topological properties. Furthermore, the relationship with the other known sets will be studied. In Section 4 we introduce the notion of generalized N−precontinuous functions.

2 PRELIMINARIES

In this section we provide some preliminary works that serve as background for the present study.

Theorem 2.1. [3] Let A and B be two subsets in a topological space (X, τ). If A is a preopen set in X and B is an open set in X then $A \cap B$ is a preopen set in X.

Theorem 2.2. [3] Let A and Y be two subsets in a topological space (X, τ). If A is a preopen set X and Y is open set in X then $A \cap Y$ is a preopen set in $(Y, \tau|Y)$.

Theorem 2.3. [3] Let Y be an open subset of a topological space (X, τ). If A is a preopen set in $(Y, \tau|Y)$ then $A = G \cap Y$ for some a preopen set G in X.

Theorem 2.4. [7] The union of any family of N−preopen sets is N−preopen set .

A subset A of topological space (X, τ) is called a *dense* in X if Cl(A) = X. A topological space (X, τ) is called submaximal space if every dense subset of X is open set.

Theorem 2.5. [7] Let (X, τ) be a submaximal space. Then $[X, NPO(X, \tau)]$ is a topological space, where $NPO(X, \tau)$ is the set of all N−preopen sets X.

Definition 2.6. A topological space (X, τ) is called:

1. T1/2−space [2] if every g−closed set is closed set.
2. T1−space [1] if for each disjoint point $x \neq y \in X$, there are two open sets G and H in X such that $x \in H$, $y \in G$, $x \notin G$ and $y \notin H$.

Theorem 2.7. [6] A topological space (X, τ) is T1/2−space if and only if every singleton set is open or closed set.

Theorem 2.8 [1] A topological space (X, τ) is T_1−space if and only if every singleton set is closed set.

Definition 2.9. A function $f : (X, \tau) \to (Y, \rho)$ of a topological space (X, τ) into a topological space (Y, ρ) is called:

1. *precontinuous* function [3] if $f^{-1}(U)$ is a preopen set in X for every open set U in Y .
2. *generalized precontinuous* function (simply g−precontinuous function) [5], if $f^{-1}(U)$ is a g−preopen set in X for every open set U in Y .
3. *N−precontinuous* function [7] if $f^{-1}(U)$ is a N−preopen set in X for every open set U in Y .
4. *generalized continuous* function(simply g−continuous function) [6] if $f^{-1}(U)$ is a g−open set in X for every open set U in Y .

Theorem 2.10. [3] Every continuous function is precontinuous function

Theorem 2.11. [3] A function $f : (X, \tau) \to (Y, \rho)$ is a precontinuous function if and only if for each $x \in X$ and each open set U in Y with $f(x) \in U$, there exists a preopen set V in X such that $x \in V$ and $f(V) \subseteq U$.

Theorem 2.12. [5] Every precontinuous function is g−precontinuous function.

Theorem 2.13. [7] Every precontinuous function is N−precontinuous function.

Theorem 2.14. [7] A function $f : (X, \tau) \to (Y, \rho)$ is a N−precontinuous function if and only if for each $x \in X$ and each open set U in Y with $f(x) \in U$, there exists a N−preopen set V in X such that $x \in V$ and $f(V) \subseteq U$.

3 Ng−PREOPEN SETS

Definition 3.1. A subset A of a topological space (X, τ) is called *generalize N−preclosed* set (simply Ng−preclosed) if $Cln(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of (X, τ). The complement of Ng−preclosed set is called generalized N−preopen set (simply Ng−preopen).

Theorem 3.2. Every N−preclosed set is Ng−preclosed set.

The proof follows immediately from the definitions and the fact Cln(A) = A if A is a N−preclosed. However, The converse of the last theorem need not be true in general as the following example shows.

Example 3.3. In topological space (N, T),

$$N = \{1, 2, 3, 4, ...\},\ T = \{ \emptyset \} \cup \{En : n \in N\},\ En = \{n, n + 1, n + 2, ...\},$$

the set N − {5} is Ng−preclosed set, since the only open set containing N − {5} is N. And N − {5} is not N−preclosed set, since there is no a finite preopen subset of N containing 5. Let U5 be a preopen set in N containing 5 such that U5 −{5} is a finite set. Then U5 will be a finite set in N. Since U5 is a preopen set in N, then

$$U5 \subseteq Int(Cl(U5) = Int[\{1, 2, 3, ..., Max(U5)\}] = \emptyset$$

and this is contradiction.

Theorem 3.4. Let (X, τ) be a topological space. If (X, τ) is a T1/2−space then every Ng−preclosed set in X is N−preclosed.

Proof. Let A be a Ng−preclosed set in X. Suppose that A is not N−preclosed set. Then there is at least $x \in Cln(A)$ such that $x \notin A$. Since (X, τ) is a T1/2−space then by Theorem(2.7), {x} is an open or closed set in X. If {x} is a closed set in X then X −{x} is an open. Since $x \notin A$ then $A \subseteq X - \{x\}$. Since A is a Ng−preclosed set and X − {x} is an open subset of X containing A, then $Cln(A) \subseteq X - \{x\}$. Hence $x \in X - Cln(A)$ and this a contradiction, since $x \in Cln(A)$. If {x} is an open set then it is N−preopen set. Since $x \in Cln(A)$ then we have $\{x\} \cap A \neq \emptyset$ That is, $x \in A$ and this a contradiction. Hence A is a N−preclosed set in X. □

It is clear that every preopen set is a N−preopen set, so the proof of the following theorem is easy, since $Cln(A) \subseteq Clp(A)$.

Theorem 3.5. Every g−preclosed set is Ng−preclosed set.

The converse of the last theorem need not be true in general as the following example shows.

Example 3.6. In topological space (X, τ), X = {a, b, c}, T = { Ø,X, {a}}, the set A = {a} is Ng−preclosed set and A is not g−preclosed set, since A is an open set in X and $A \subseteq A$ but $Cl(A) = X \not\subset A$.

We have the following relation for Ng−preopen set with the other known sets.

open	→	preopen	→	N-preopen
↓		↓		↓
g-open	→	g-preopen	→	Ng-preopen

Figure 1

Lemma 3.7. For a topological space (X, τ) and $A \subseteq X$, the following hold:
1. $Intn(X - A) = X - Cln(A)$.

2. $Cln(X - A) = X - Intn(A)$.

Proof. 1. Since Cln(A) is a N−preclosed set and $A \subseteq Cln(A)$, then

$$X - Cln(A) \subseteq X - A,$$

this implies

$$X - Cln(A) = Intn[X - Cln(A)] \subseteq Intn(X - A).$$

For the other side, let $x \in Intn(X - A)$. Then there is N−preopen set U such that $x \in U \subseteq X - A$. Then $X - U$ is a N−preclosed set containing A and $x \notin X - U$. Hence $x \notin Cln(A)$, that is, $x \in X - Cln(A)$.
2. Similar for the Part (1). □

Theorem 3.8. A subset A of a topological space (X, τ) is a Ng−preopen if and only if $F \subseteq Intn(A)$ whenever $F \subseteq A$ and F is closed subset of (X, τ).

Proof. Let A be a Ng−preopen subset of X and F be a closed subset of X such that $F \subseteq A$. Then $X - A$ is a Ng−preclosed, $X - A \subseteq X - F$ and $X - F$ is an open subset of X. Hence Lemma(3.7), $X - Intn(A) = Cln(X - A) \subseteq X - F$, that is, $F \subseteq Intn(A)$.
Conversely, suppose that $F \subseteq Intn(A)$ where F is a closed subset of X such that $F \subseteq A$. Then for any open subset U of X such that $X - A \subseteq U$, we have $X - U \subseteq A$ and $X - U \subseteq Intn(A)$. Then by Lemma(3.7), $X - Intn(A) = Cln(X - A) \subseteq U$. Hence $X - A$ is a Ng−preclosed (i.e., A is a Ng−preopen set). □

Theorem 3.9. If A is a Ng−preclosed subset of a topological space (X, τ) then $Cln(A)-A$ contains no nonempty closed set.

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Proof. Suppose that $Cln(A) - A$ contains nonempty closed set F. Then

$$F \subseteq Cln(A) - A \subseteq Cln(A).$$

Since $A \subseteq Cln(A)$ then $F \subseteq X - A$ and so $A \subseteq X - F$. Since A is a Ng−preclosed set and $X - F$ is an open subset of X, then $Cln(A) \subseteq X - F$ and so $F \subseteq X - Cln(A)$. Therefore $F \subseteq Cln(A) \bigcap (X - Cln(A)) = \varnothing$ and so $F = \varnothing$ Hence $Cln(A) - A$ contains no nonempty closed set. □

Corollary 3.10. If A is a Ng−preclosed subset of a topological space (X, τ) then $Cln(A)-A$ is a Ng−preopen set.

Proof. By Theorem(3.9), $Cln(A) - A$ contains no nonempty closed set and it is clear that $\varnothing \subseteq Intn(Cln(A) - A)$ then $Cln(A) - A$ is a Ng−preopen set. □

Theorem 3.11. If A is a Ng−preclosed subset of a topological space (X, τ) and $B \subseteq X$. If $A \subseteq B \subseteq Cln(A)$ then B is a Ng−preclosed set.

Proof. Let U be an open set in X such that $B \subseteq U$. Then $A \subseteq B \subseteq U$. Since A is a

Ng−preclosed set then $Cln(A) \subseteq U$. Since $B \subseteq Cln(A)$ then

$$Cln(B) \subseteq Cln[Cln(A)] = Cln(A) \subseteq U.$$

That is, B is a Ng−preclosed set. □

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Theorem 3.12. Let A be a Ng−preclosed subset of a topological space (X, τ). Then $A = Cln(Intn(A))$ if and only if $Cln(Intn(A)) - A$ is a closed set.

Proof. Let $Cln(Intn(A)) - A$ be a closed set. Since $Intn(A) \subseteq A$ and $A \subseteq Cln(A)$, then

$$Cln(Intn(A)) \subseteq Cln(A). \text{ Then } Cln(Intn(A)) - A \subseteq Cln(A) - A,$$

this implies

$$Cln(Intn(A)) - A \subseteq X - A \Rightarrow A \subseteq X - (Cln(Intn(A)) - A).$$

Since A is a Ng−preclosed set and $X - (Cln(Intn(A)) - A)$ is an open set containing A, then $Cln(A) \subseteq X - (Cln(Intn(A)) - A)$, this implies

$$Cln(Intn(A)) - A \subseteq X - Cln(A).$$

Therefore

$$Cln(Intn(A)) - A \subseteq Cln(A) \bigcap (X - Cln(A)) = \varnothing .$$

Hence $Cln(Intn(A)) - A = \varnothing$ that is, $Cln(Intn(A)) = A$.

Conversely, if $A = Cln(Intn(A))$ then $Cln(Intn(A)) - A = \varnothing$ and hence $Cln(Intn(A)) - A$ is a closed set. □

Lemma 3.13 For a topological space (X, τ) and $A \subseteq X$, $x \in Cln(A)$ if and only if for all N−preopen set U containing x, $U \bigcap A \neq \varnothing$

Proof. Let $x \in Cln(A)$ and U be a N−preopen set containing x. If $U \bigcap A = \varnothing$ then $A \subseteq X - U$. Since $X - U$ is a N−preclosed set containing A, then $Cln(A) \subseteq X - U$ and so $x \in Cln(A) \subseteq X - U$. Hence this is contradiction, because $x \in U$. Therefore $U \bigcap A \neq \varnothing$.

Conversely, Let $x \notin Cln(A)$. Then $X - Cln(A)$ is a N−preopen set containing x. Hence by hypothesis, $[X - Cln(A)] \bigcap A \neq \varnothing$. But this is contradiction, because $X - Cln(A) \subseteq X - A$.

□

Lemma 3.14. Let Y be an open subset of a topological space (X, τ). Then the following hold:

1. If A is a N−preopen set in (X, τ) then $A \bigcap Y$ is a N−preopen set in $(Y, T|Y)$.
2. If A is a N−preclosed set in $(Y, T|Y)$ then A is a N−preclosed set in (X, τ).
3. If A is a N−preopen set in $(Y, T|Y)$ then A is N−preopen set in (X, τ).
4. If $A \subseteq Y$ then $Cln|Y (A) = Cln(A) \bigcap Y$.

Proof. 1. Let A be a N−preopen set in (X, τ) and $x \in A \bigcap Y$. This implies $x \in A$ and $x \in Y$. Hence there is a preopen set U in X containing x such that $U - A$ is a finite. $x \in Y$ and by Theorem(2.2), the set $U \bigcap Y$ is a preopen in $(Y, T|Y)$ containing x and

$$(U \cap Y) \cap (Y - (A \cap Y)) = (U \cap Y) \cap (Y \cap (X - A))$$
$$= U \cap (X - A) \cap Y$$
$$= (U - A) \cap Y.$$

Since $U - A$ is a finite, then $(U - A) \cap Y$ is a finite. That is, $A \cap Y$ is a N−preopen set in $(Y, T|Y)$.
2. Let $x \in Y - A$. Since A is a N−preclosed set $(Y, T|Y)$, then there is a preopen set U in $(Y, T|Y)$ containing x such that $U \cap A = U \cap [Y - (Y - A)]$ is a finite. Since U is a preopen in $(Y, T|Y)$, then by Theorem (2.3), $U = O \cap Y$ for some preopen set O in X. Since Y is an open set in X and O is a preopen set in X, then by Theorem (2.1), $U = O \cap Y$ is preopen set in X containing x. Hence $Y - A$ is a N−preopen set in (X, τ), that is, A is a N−preclosed set in (X, τ).
3. Similar for the part(2).
4. Let $x \in Cln|Y(A)$ and G be a N−preopen set in X containing x. By part(1), $G \cap Y$ is a N−preopen set in Y containing x and since $x \in Cln|Y(A)$, then

$$G \cap A = (G \cap Y) \cap A \neq \varnothing.$$

Hence by Lemma(3.13), $x \in Cln(A)$, and since $x \in Y$, this implies $x \in Cln(A) \cap Y$. That is, $Cln|Y(A) \subseteq Cln(A) \cap Y$. On the other side, let $x \in Cln(A) \cap Y$ and O be a N−preopen set in Y containing x. By part(3), $O = G \cap Y$ for some N−preopen set G in X. Since $x \in Cln(A)$, then $G \cap A \neq \varnothing$ and so $(G \cap Y) \cap A \neq \varnothing$, since $x \in Y$ Hence $O \cap A \neq \varnothing$ that is, $x \in Cln|Y(A)$. Hence $Cln(A) \cap Y \subseteq Cln|Y(A)$. □

Theorem 3.15. Let Y be an open subspace of a topological space (X, τ) and $A \subseteq Y$. If A is a Ng−preclosed subset in X then A is a Ng−preclosed set in Y.

Proof. Let O be an open subset in Y such that $A \subseteq O$. Then $O = U \cap Y$ for some open set U in X and so $A \subseteq U$. Since A is a Ng−preclosed subset of X, then $Cln(A) \subseteq U$. By Lemma(3.14), $Cln|Y(A) = Cln(A) \cap Y \subseteq U \cap Y = O$. Hence A is a Ng−preclosed set in Y. □

Theorem 3.16. Let Y be an open subspace of a topological space (X, τ) and $A \subseteq Y$. If A is a Ng−preclosed subset in Y and Y is N−preclosed in X then A is a Ng−preclosed set in X.

Proof. Let U be an open subset in X such that $A \subseteq U$. Then $A \subseteq U \cap Y$ and $U \cap Y$ is open set in Y. Since A is a Ng−preclosed subset in Y, then $Cln|Y(A) \subseteq U \cap Y$. Since Y is an open set in X and it is N−preclosed in X then

$$Cln(A) = Cln(A \cap Y) \subseteq Cln(A) \cap Cln(Y) = Cln(A) \cap Y$$
$$= Cln|Y(A) \subseteq U \cap Y \subseteq U.$$

Hence A is a Ng−preclosed set in X. □

A topological space (X,τ) is called a *locally prefinite* space if for each $x\in X$, there is a finite preopen set U_x in X such that $x\in U_x$. A topological space (X,τ) is called *anti-locally prefinite* space if each nonempty preopen set in X is an infinite set.

Lemma 3.17. Let $(Y, T|Y)$ be anti-locally prefinite subspace of (X,τ). If Y is an open set in X then $Clp(Y) = Cln(Y)$.

Proof. It is clear that $Cln(Y)\subseteq Clp(Y)$. Now we need to prove that $Clp(Y)\subseteq Cln(Y)$. Suppose that there is $x\notin Cln(Y)$ and $x\in Clp(Y)$. Since $x\notin Cln(Y)$, then there is at least one N−preopen set U containing x such that $U\cap Y=\varnothing$. Since $x\in U$ and U is a N−preopen set, choose a preopen set V containing x such that $V-U=M$ is a finite set. Since $x\in Clp(Y)$ and V is a preopen set containing x, then $V\cap Y\neq\varnothing$. Since

$$Y\cap V = Y\cap (U\cup M) = (Y\cap U)\cup (Y\cap M))$$
$$= \varnothing\cup (Y\cap M) = Y\cap M\subseteq Y\cap V.$$

Then $V\cap Y = M\cap Y$. Since Y is an open set in Y , then by Theorem(2.1), $M\cap Y$ is a preopen set in Y but $M\cap Y$ is a finite set and this contradicts the fact that $(Y, T|Y)$ be anti-locally prefinite. Hence $Clp(Y)\subseteq Cln(Y)$. □

The proof of the following theorem is clear from Lemma(3.17).

Theorem 3.18. Let $(Y, T|Y)$ be anti-locally prefinite subspace of (X,τ) and Y be an open set in X. Then Y is a Ng−preclosed set in X if and only if it is a g−preclosed set X

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Theorem 3.19. Let (X,τ) be anti-locally prefinite space. Then X is T_1- space if and only if every Ng−preclosed set is a N−preclosed set in X.

Proof. Sufficiency: Let $x\in X$ be an arbitrary point in X. By using Theorem(2.8), to prove that X is T_1- space, we will prove that $\{x\}$ is a closed set in X. Suppose that $\{x\}$ is not closed set in X. Then $A = X-\{x\}$ is not open set. Then X is the only open set containing A and hence $Cln(A)\subseteq X$, that is, A is a Ng−preclosed set in X. Then, by assumption, A is a N−preclosed set. That is, $\{x\}$ is a N−preopen set. Hence there is a preopen set V in X containing x such that $V-\{x\}$ is a finite set. It follows that V is a nonempty finite preopen set in X contradicts the fact (X,τ) be anti-locally prefinite space. Then X is T_1- space.
Necessity: By Theorem(2.8) and Theorem(2.7), it is clear that X is a $T_{1/2}-$ space. Then, by Theorem(3.4), every Ng−preclosed set is a N−preclosed set in X. □

Theorem 3.20. If A is a Ng−preclosed set in a topological space (X,τ) and B is a closed set in X then $A\cap B$ is a Ng−preclosed set.

Proof. Let U be an open subset of X such that $A \cap B \subseteq U$. Since B is a closed set in X then $U \cup (X - B)$ is an open set in X. Since A is a Ng−preclosed set in X and $A \subseteq U \cup (X - B)$ then $Cln(A) \subseteq U \cup (X - B)$. Hence

$$\begin{aligned} Cln(A \cap B) &\subseteq Cln(A) \cap Cln(B) \subseteq Cln(A) \cap Cl(B) \\ &= Cln(A) \cap B \subseteq [U \cup (X - B)] \cap B \\ &\subseteq U \cap B \subseteq U. \end{aligned}$$

Thus, $A \cap B$ is a Ng−preclosed set. □

4 Ng −PRECONTINUOUS FUNCTIONS

Definition 4.1. A function $f : (X, \tau) \to (Y, \rho)$ of a topological space (X, τ) into a space (Y, ρ) is called *generalized N−precontinuous* (simply Ng−precontinuous) function, if $f^{-1}(U)$ is a Ng−preopen set in X for every open set U in Y .

Theorem 4.2. A function $f : (X, \tau) \to (Y, \rho)$ of a topological space (X, τ) into a space (Y, ρ) is Ng−precontinuous if and only if $f^{-1}(F)$ is a Ng−preclosed set in X for every closed set F in Y .

Proof. Let $f : (X, \tau) \to (Y, \rho)$ be a Ng−precontinuous and F be any closed set in Y . Then $f^{-1}(Y - F) = X - f^{-1}(F)$ is a Ng−preopen set in X, that is, $f^{-1}(F)$ is Ng−preclosed set in X.
Conversely, suppose that $f^{-1}(F)$ is a Ng−preclosed set in X for every closed set F in Y . Let U be any open set in Y . Then by the hypothesis, $f^{-1}(Y - U) = X - f^{-1}(U)$ is a Ng−preclosed set in X, that is, $f^{-1}(U$ is a Ng−preopen set in X. Hence f is a Ng−precontinuous. □

It is clear that every N−precontinuous function is Ng−precontinuous and the converse need not be true in general.

Example 4.3. Let $f : (N, \tau) \to (Y, \rho)$ be a function defined by

$$f(n) = \begin{cases} a, & n = 5 \\ b, & n \neq 5 \end{cases}$$

where
$N = \{1, 2, 3, 4, ...\}$, $T = \{\emptyset\} \cup \{En : n \in N\}$, $En = \{n, n + 1, n + 2, ...\}$,
$Y = \{a, b\}$ and $\rho = \{\emptyset, Y, \{a\}\}$. The function f is a Ng−precontinuous, since $f^{-1}(\{a\}) = \{5\}$ and $f^{-1}(Y) = N$ are Ng−preopen sets in N. The function f is not N−precontinuous, see Example (3.3), $f^{-1}(\{a\}) = \{5\}$ is not N−preopen set in N.

Theorem 4.4. Let $f : (X, \tau) \to (Y, \rho)$ be a function of a T1/2−space (X, τ) into a space(Y, ρ). If f is a Ng−precontinuous then it is a N−precontinuous.

Proof. Let $f : (X, \tau) \to (Y, \rho)$ be a Ng−precontinuous function and U be any open set Y . Then $f^{-1}(U)$ is a Ng−preopen set in X. Since X is a T1/2−space then by Theorem(3.4), $f^{-1}(U)$ is a N−preopen set in X. That is, f is a N−precontinuous function. □

It is clear that every g−precontinuous function is Ng−precontinuous and the converse need not be true.

Example 4.5. Let $f : (X, \tau) \to (Y, \rho)$ be a function defined by f(a) = f(c) = 1 and f(b) = 2 where $X = \{a, b, c\}$, $Y = \{1, 2\}$, $\tau = \{\varnothing, X, \{a, b\}\}$ and $\rho = \{\varnothing, Y, \{1\}\}$. The function f is a Ng−precontinuous. The set $f^{-1}(\{1\}) = \{a, c\}$ is not g−preopen set in X, since $X - \{a, c\} = \{b\} \subseteq \{a, b\}$ but

$$Clp(X - \{a, c\}) = Clp(\{b\}) = X \not\subset \{a, b\},$$

that is, the function f is not g−precontinuous.

Lemma 4.6. Let $f : (X, \tau) \to (Y, \rho)$. be a function of an anti-locally prefinite submaximal space (X, τ) onto a regular space (Y, ρ). Then the following are equivalent:

1. f is continuous.
2. f is precontinuous.
3. f is N−precontinuous.

Proof. 1 $\Rightarrow$ 2: By Theorem(2.10).
2 $\Rightarrow$ 3: By Theorem(2.13).
3 $\Rightarrow$ 1: Let $x \in X$ be an arbitrary point in X and V be an open set in Y such that $f(x) \in V$. By regularity of Y , there is an open set W in Y such that

$$f(x) \in W \subseteq ClY(W) \subseteq V$$

.Since f is N−precontinuous and W is open set in Y containing f(x), then by Theorem(2.14), there is a N−preopen set F in X containing x such that $f(F) \subseteq W$. Then there is a preopen set G in X containing x such that $G - F$ is a finite set. We claim $f(G) \subseteq ClY(W)$. If not, there is at least $y \in f(G)$ and $y \notin ClY(W)$. Therefore $y = f(g)$ for some $g \in G$. Now we observe that $y \in Y - ClY(W)$ and $Y - ClY(W)$ is an open set in Y . Then, since f is N−precontinuous and by Theorem(2.14) again, there is a N−preopen set U in X containing g such that $f(U) \subseteq Y - ClY(W)$. Then there is a preopen set H in X containing x such that $H - U$ is a finite set. Hence

$$f[F] \cap f(U) \subseteq W \cap [Y - ClY(W)] \subseteq ClY(W) \cap [Y - ClY(W)] = \varnothing.$$

Hence $F \cap U = \varnothing$ and $g \in G \cap H \subseteq (G - F) \cup (U - H)$. That is, $G \cap H$ is a finite set. Since X is a submaximal then by Theorem (2.5), G and H are open sets in X and so $G \cap H$ is a preopen finite set in X, which contradicts the fact that X is an anti-locally prefinite. Therefore $f(G) \subseteq ClY(W) \subseteq V$, that is, f is a continuous function. □

Theorem 4.7.Let $f : (X, \tau) \to (Y, \rho)$ be a function of an anti-locally prefinite submaximal

T1/2−space (X, τ) onto a regular space (Y, ρ). Then the following are equivalent:
1. f is continuous.
2. f is g−precontinuous.
3. f is Ng−precontinuous.

Proof. $1 \Rightarrow 2$: By Theorem(2.10).
$2 \Rightarrow 3$: Trivial.
$3 \Rightarrow 1$: By Theorem(4.4) and Lemma (4.6). □

We have the following relation for Ng−precontinuous function with the other known functions.

continuous	→	precontinuous	→	N-precontinuous
↓		↓		↓
g-continuous	→	g-precontinuous	→	Ng-precontinuous

Figure 2

Theorem 4.8. If $f : (X, \tau) \rightarrow (Y, \rho)$ is a Ng−precontinuous function then for each $x \in X$ and each open set U in Y with $f(x) \in U$, there exists a Ng−preopen set V in X such that $x \in V$ and $f(V) \subseteq U$.

Proof. Let $x \in X$ and U be any open set in Y containing f(x). Put $V = f^{-1}(U)$. Since f is a Ng−precontinuous then V is a Ng−preopen set in X such that $x \in V$ and $f(V) \subseteq U$. □

The converse of the last theorem need not be true.

Example 4.9. Let $f : (N, T) \rightarrow (Y, \rho)$ be a function defined by

$$f(n) = f(x) = \begin{cases} a, & n \in N - E6 \\ b, & n \in E6 \end{cases}$$

where

$$T = \{\emptyset, N\} [\{En : n \in N \text{ and } n \geq 6\}, En = \{n, n+1, n+2, ...\},$$

$Y=\{a, b\}$ and $\rho = \{\emptyset, Y, \{a\}\}$. The function f is not a Ng−precontinuous, $f^{-1}(\{a\}) = N - E6$ is not Ng−preopen set in N. On the other hand, for each $n \in N$ and each open set U in Y containing f(n), the set $V = \{n\}$ is a Ng−preopen set in N containing n and $f(V) \subseteq U$.

The proof of the following lemma is similar for the proof of Theorem(4.2).

Lemma 4.10. A function $f : (X, \tau) \rightarrow (Y, \rho)$ of a topological space (X, τ) into a space (Y, ρ) is N−precontinuous if and only if $f^{-1}(F)$ is a N−preclosed set in X for every closed set F in Y .

Theorem 4.11. Let $f : (X, \tau) \to (Y, \rho)$ be a function of a T1/2−space (X, τ) into a space (Y, ρ). Then the following are equivalent:

1. f is N−precontinuous.
2. $f[Cl_n^X(A)] \subseteq Cl^Y(f(A))$ for all $A \subseteq X$.
3. f is Ng−precontinuous.

Proof. 1 $\Rightarrow$ 2: Let A be any subset of X. Then $Cl^Y(f(A))$ is a closed set in Y . Since f is a N−precontinuous then by Lemma(4.10), $f^{-1}[Cl^Y(f(A))]$ is a N−preclosed set in X. That is,

$$Cl_n^X(f^{-1}[Cl^Y(f(A))]) = f^{-1}[Cl^Y(f(A))].$$

Since $f(A) \subseteq Cl^Y(f(A))$ then $A \subseteq f^{-1}[Cl^Y(f(A))]$. This implies,

$$Cl_n^X(A) \subseteq Cl_n^X(f^{-1}[Cl^Y(f(A))]) = f^{-1}[Cl^Y(f(A))].$$

Hence $f[Cl_n^X(A)] \subseteq Cl^Y(f(A))$.

2 $\Rightarrow$ 3: Let H be any closed set in Y , that is, $Cl^Y(H) = H$. Since $f^{-1}(H) \subseteq X$. Then by the hypothesis,

$$f(Cl_n^X[f^{-1}(H)]) \subseteq Cl^Y[f(f^{-1}(H))] \subseteq Cl^Y(H) = H.$$

This implies, $Cl_n^X[f^{-1}(H)] \subseteq f^{-1}(H)$. Hence $Cl_n^X[f^{-1}(H)] = f^{-1}(H)$, that is, $f^{-1}(H)$ is a N−preclosed set in X. Hence $f^{-1}(H)$ is a Ng−preclosed set in X. That is, f is a Ng−precontinuous.

3 $\Rightarrow$ 1: Since (X, τ) is a T1/2−space then by Theorem(4.4), f is N−precontinuous. □

Theorem 4.12. Let $f : (X, \tau) \to (Y, \rho)$ be a function of a T1/2−space (X, τ) into a space (Y, ρ). Then the following are equivalent:

1. f is N−precontinuous.
2. $Cl_n^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))$ for all $B \subseteq Y$.
3. f is Ng−precontinuous.

Proof. 1 $\Rightarrow$ 2: Let B be any subset of Y . Then $Cl^Y(B)$ is a closed set in Y . Since f is a N−precontinuous then by Lemma(4.10), $f^{-1}[Cl^Y(B)]$ is a N−preclosed set in X. That is,

$$Cl_n^X([Cl^Y(B)] = f^{-1}[Cl^Y(B)].$$

Since $B \subseteq Cl^Y(B)$ then $f^{-1}(B) \subseteq f^{-1}[Cl^Y(B)]$. This implies,

$$Cl_n^X(f^{-1}(B)) \subseteq Cl_n^X(f^{-1}[Cl^Y(B)]) = f^{-1}[Cl^Y(B)].$$

Hence $Cl_n^X(f^{-1}(B)) \subseteq f^{-1}[Cl^Y(B)]$.

2 $\Rightarrow$ 3: Let H be any closed set in Y , that is, $Cl^Y(H) = H$. Since $H \subseteq Y$. Then by the hypothesis,

$$Cl_n^X(f^{-1}(H)) \subseteq f^{-1}(Cl^Y(H)) = f^{-1}(H).$$

This implies, Cl_n^X $[f^{-1}(H)] \subseteq f^{-1}(H)$. Hence Cl_n^X $[f^{-1}(H)] = f^{-1}(H)$, that is, $f^{-1}(H)$ is a N−preclosed set in X. Hence $f^{-1}(H)$ is a Ng−preclosed set in X. That is, f is a Ng−precontinuous.

3 $\Longrightarrow$ 1: Since (X, τ) is a T1/2−space then by Theorem(4.4), f is N−precontinuous. □

Theorem 4.13. Let f : (X, τ) $\longrightarrow$ (Y, ρ) be a function of a T1/2−space (X, τ) into a space (Y, ρ). Then the following are equivalent:

1. f is N−precontinuous.
2. $f^{-1}(\text{Int}^Y(B)) \subseteq \text{Int}_n^X[f^{-1}(B)]$ for all $B \subseteq Y$.
3. f is Ng−precontinuous.

Proof. 1$\Longrightarrow$ 2: Let B be any subset of Y . Then Int^Y (B) is an open set in Y . Since f is a N−precontinuous then $f^{-1}[\text{Int}^Y(B)]$ is a N−preopen set in X. That is,

$$\text{Int}_n^X\{f^{-1}[\text{Int}^Y(B)]\} = f^{-1}[\text{Int}^Y(B)].$$

Since $\text{Int}^Y(B) \subseteq B$ then $f^{-1}[\text{Int}^Y(B)] \subseteq f^{-1}(B)$. This implies,

$$f^{-1}[\text{Int}^Y(B)] = \text{Int}_n^X\{f^{-1}[\text{Int}^Y(B)]\} \subseteq \text{Int}_n^X(f^{-1}(B)).$$

Hence $f^{-1}(\text{Int}^Y(B)) \subseteq \text{Int}_n^X[f^{-1}(B)]$.

2$\Longrightarrow$ 3: Let U be any open set in Y , that is, $\text{Int}^Y(U) = U$. Since $U \subseteq Y$. Then by the hypothesis,

$$f^{-1}(U) = f^{-1}(\text{Int}^Y(U)) \subseteq \text{Int}_n^X[f^{-1}(U)].$$

This implies, $f^{-1}(U) \subseteq \text{Int}_n^X[f^{-1}(U)]$. Hence $f^{-1}(U) = \text{Int}_n^X[f^{-1}(U)]$, that is, $f^{-1}(U)$ is a N−preopen set in X. Hence by Theorem(3.2), $f^{-1}(U)$ is a Ng−preopen set in X. That is, f is a Ng−precontinuous.

3$\Longrightarrow$1: Since (X, τ) is a T1/2−space then by Theorem(4.4), f is N−precontinuous. □

Theorem 4.14. If f : (X, τ) $\longrightarrow$ (Y, ρ) is a Ng−precontinuous function and A is an open subspace of topological space (X, τ) then the restriction function f|A : (A, τ A) $\longrightarrow$ (Y, ρ) of f on A is a Ng−precontinuous.

Proof. Let U be an open set in Y . since f is a Ng−precontinuous then $f^{-1}(U)$ is a Ng−preopen set in X. Since A is an open in X then A is a Ng−preopen set in X. Then $f^{-1}(U) \cap A = (f|A)^{-1}(U)$ is a Ng−preopen set in X. Then by Theorem(3.15), $((f|A)^{-1}(U) \subseteq A$ is a Ng−preopen set in A. That is, f|A is a Ng−precontinuous. □

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عائلة المجموعات المفتوحة (N−preopen)

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ملخص
الغرض الاساسي من هذا البحث هو تقديم عائلة جديدة جزئية من عائلة المجموعات المفتوحة (N−preopen) تسمى (Ng−preopen) ودراسة الخصائص التبولوجية على هذه العائلة وعلاقتها بالعوائل الاخرى. بالإضافة الى تقديم ودراسة الاستمرارية للدوال بدلالة عائلة المجموعات (Ng−preopen).

كلمات مفتاحية: N−preopen ، Ng−preopen.