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Original Article

Allee Rate Effect in a Prey Model with a Holling Type II Functional Response

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Abstract:

In this work, we have built a detailed model of a prey-predator system with mechanical Allee rate and Hollings type II functional response. The model's positive accumulation points' existence, uniqueness, boundaries, and stability analysis are taken into consideration. Numerical simulations are used to discuss Allee's impact on system dynamics.

Keywords: Prey Predator; Allee Effect; Halling Type II; Local Stability; Lyapunov Function

1. Introduction

Population dynamics is one of the most popular topics in biomathematics. A particular area of interest has always been the evolution of different societies, from aggregations of single species to more realistic representations of the coexistence and interaction of numerous species within the same ecosystem. One of the most important and well-researched models for the interaction of living organisms is the prey-predator paradigm. Vito Volterra and Alfred J. Lotka were the first to introduce this approach in 1926 and 1925 respectively [1]. Since Allee and other scientists have studied these models, it has been possible to use systems of ordinary or partial differential equations to represent most predator-prey interactions.

Allee discovered that a person's social desirability increases population growth, which increases density and consequently competition for resources [2]. This study was among the best studies that obtained benefits for population centers by concentrating resources in one place so that growth is conditional. The predator-prey paradigm of the avenue effect on prey growth has also attracted considerable interest. [3, 4, 5, 6, 7, 8, 9]. Longxing Qi and Lijuan Gan also investigated Allee's effects on prey with shelter [10].

In this research, we built a model consisting of a prey-predator, and it contains a response function of type II and Allee strong rate, and the study showed the effect of the Allee rate in the model.

The Halling type II functional response in predator development:

$$\frac{dx}{dt} = xh(x) - yB(x)$$
$$\frac{dy}{dt} = yB(x) - f(y)dy$$

where $h(x) = r\left(1-\frac{x}{k}\right)(x-b)$ and $B(x) = \frac{ax}{1+x}$ The beginning circumstances are x(0) and y(0) > 0. The prey population is represented by x, The predator population is represented by y, the predator mortality rate is represented by f(y), and the conversion efficiency of prey to predator is represented by c. K carrying capacity, h(x) per capita prey growth rate, r prey growth rate specific, b Allee effect threshold, B(x) prey-dependent functional response and a maximal attack rate are all represented. hence, we do:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right)(x-b) - \frac{axy}{1+x} \quad ,$$
$$\frac{dy}{dt} = \frac{cxy}{1+x} - dy \quad ,$$

all parameters are positive, the average predator loss rate is d.

2. Existence of accumulation point

In this part, we will just look at the system's coexisting accumulation point (1). If the following set of equations has a positive solution, a positive accumulation point $E_1 = (x*, y*)$ exists and is unique within int. R_+^2 in xy space:

$$r\left(1-\frac{x}{k}\right)(x-b) - \frac{ay}{1+x} = 0 \ , \quad (1)$$
$$\frac{cx}{1+x} - d = 0 \quad , \quad (2)$$

Here $* = \frac{d}{c-d}$, $y* = \frac{1+x*}{a}\left(r\left(1-\frac{x*}{k}\right)(x-b)\right)$, For a positive accumulation point, we have $b < x* < k$.

3. Boundedness of the model

Theorem (1): All of the system (1) solutions that start at R_+^2 are uniformly bounded.

Proof.

Assume that $(x(t), y(t))$ be any solution to the system (1) with the initial conditions (x(0), y(0)) being non-negative, describe the function:

$$M(t) = x(t) + \frac{a}{c}y(t) \ . \quad (3)$$

Therefore, we are derivative equation (3),

$\frac{dM}{dt} = rx\left(1-\frac{x}{k}\right)(x-b) - \frac{axy}{1+x} + \frac{a}{c}\left(\frac{cxy}{1+x} - dy\right)$,
Now $0 \le x \le 1$, we have:

$\frac{dM}{dt} < r\left(1+\frac{b}{k}\right)x^2 - brx - \frac{ad}{c}y$,
$\frac{dM}{dt} < r\left(1+\frac{b}{k}\right) - brx - \frac{ad}{c}y$,
$\frac{dM}{dt} = H - n\,M.$

where $n = \min\left\{br, \frac{ad}{c}\right\}$, and $H = r\left(1+\frac{b}{k}\right)$

$$M(t) \le \frac{H}{n} + \left(M(0) - \frac{2}{n}\right) e^{-nt}.$$

Thus $0 \le M(t) \le \frac{2}{n}$,as $t \to \infty$. As a result, the proof is successful since all solutions to system (1) are uniformly bounded.

4. Local Stability Analysis

In this section, we study the local stability of the model (1) around positive accumulation points, and it is fairly simple to construct the Jacobian matrix J(x,y) of the system (1) by computing the Jacobian matrix J(x,y) and the eigenvalues of system (1) at each of them.

$$J_1 = J(E) = \begin{pmatrix} f_1 f_2 \\ f_3 f_4 \end{pmatrix}$$

where:

$$f_1 = -2\frac{r}{k}x*^2 + rx*\left(1+\frac{b}{k}\right) + \frac{ax*y*}{(1+x*)^2}, \quad f_2 = \frac{-ax*}{1+x*}$$

$$f_3 = \frac{cy*}{(1+x*)^2}, \quad f_4 = 0$$

Then the distinctive equation of $J(E_1)$ is given by:

$\lambda^2 + U_1\lambda + U_2 = 0$, where $U_1 = -f_1, U_2 = -f_2 f_3$,

Thus, we have the following conclusions:

a) If $U_1 < 0$ (i.e. $2\frac{r}{k}x* > r\left(1+\frac{b}{k}\right) + \frac{ay*}{(1+x*)^2}$), then the positive accumulation is locally asymptotically stable.

b) If $U_1 > 0$ (i.e. $2\frac{r}{k}x* < r\left(1+\frac{b}{k}\right) + \frac{ay*}{(1+x*)^2}$), then the positive accumulation is unstable.

5. Global stability

Theorem: The following conditions apply: $s_1 < s_2$, $0 \le x \le 1$ and E is locally asymptotically stable. In this case E is globally asymptotically stable.

Thus,

$s_1 = r(x-x*)\left(\left(1-\frac{x}{k}\right)(x-b) - \left(1-\frac{x*}{k}\right)(x*-b)\right)$, $s_2 = ray*(x-x*)(1-x*)$,

Proof:
Consider the following function:

$G(x,y) = \left(x - x* - x*\, ln\frac{x}{x*}\right) + \frac{ra}{c}\left(y - y* - y*\, ln\frac{y}{y*}\right)$.
$G(x,y) \epsilon\, C^1(R_+^2, R)$, $G(E) = 0$, and $G(x,y) > 0$;

$\forall(x,y) \ne E$. Now differentiate G from time t onwards and do some algebraic work taking this into account:

$$\frac{dG}{dt} = r(x-x*)\left(\left(1-\frac{x}{k}\right)(x-b) - \frac{ay}{1+x} - \left(1-\frac{x*}{k}\right)(x*-b) + \frac{ay*}{1+x*}\right) + ra(y-y*)\left(\frac{x}{1+x} - \frac{x*}{1+x*}\right)$$

$$\frac{dG}{dt} \le r(x-x*)\left(\left(1-\frac{x}{k}\right)(x-b) - \left(1-\frac{x*}{k}\right)(x*-b)\right) - ra(x-x*)\left(\frac{y}{1+x} + \frac{y*}{1+x*}\right) + ra(y-y*)\left(\frac{x}{1+x} - \frac{x*}{1+x*}\right),$$

$$\frac{dG}{dt} \le r(x-x*)\left(\left(1-\frac{x}{k}\right)(x-b) - \left(1-\frac{x*}{k}\right)(x*-b)\right) - ray*(x-x*)(1-x*).$$

$\frac{dG}{dt} = s_1 - s_2.$

6. Numerical simulation

We numerically simulate the aforementioned theoretical reasoning in this part. by MATLAB.

6.1 Strong Allee effect

The ODE model (1) has four parameters: r ,k , b, a , c, d. We choose the parameters:

Table 1: The parameters (r, k, b, a, c, and d) of the ODE model (1).

Index	Parameter					
	r	*k*	*b*	*A*	*C*	*d*
1	0.9	0.8	0.01	0.8	0.7	0.2
2	0.9	0.8	0. 1	0.8	0.7	0.2
3	0.9	0.8	0.01	0.8	0.5	0.2

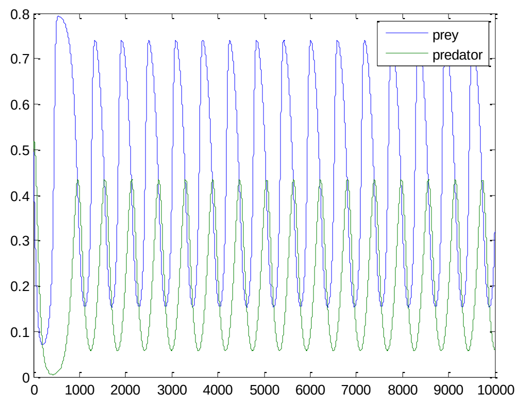

Figure 1: The periodic trajectories of the system.

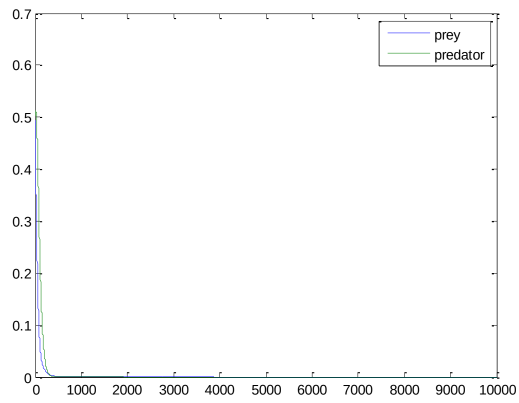

Figure 2: Trivial point of the system.

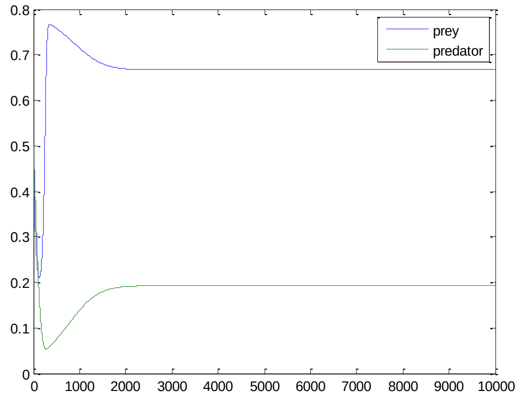

Figure 3: shows asymptotically stable of E = (x∗, y∗) = (0.666, 0.201).

According to Figure 1, we can find the system has periodic trajectories if we take set 1 in table and all conditions of local stale are hold. If we take set 2, then the system has trivial point as shown in Figure 2. If we take the set of parameters 3, then E = (x∗, y∗) = (0.666, 0.201) is asymptotically stable as shown in Figure 3.

7. Conclusions

In this article, a predator-prey paradigm is described with an Allee effect in prey growth, and a functional type II response to Holling in predator growth the system was studied in detail analytically for the positive equilibrium point. The existence of this point depends on the fulfillment of the following condition $b < x* < k$, and it is stable locally if the condition is fulfilled $U_1 < 0$, and when the conditions for existence and stability are met locally and the following condition $s_1 < s_2$, the positive point is globally stable. The study of the effect of the strong mechanism on the system made the parameter b possess characteristics that change the behavior of the system, so when we took the group (1) in the table, we observed that the system has a periodic cycle as shown in Figure 1, and we maintained that when the value of the parameter changed (b), Observing the behavior of the system changed suddenly, and we obtained the Figure 2, in both cases, all parameters were constant. While when the value of parameter (b) was constant and the value of the predation rate changed, the behavior of the system also changed, and we obtained a globally stable point. We can say, after studying the system, that the rate of change Allee plays a major role in the stability of the system.

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