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Maximal Subgroups of the Group PSL(11, 2)

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ABSTRACT

In this note, we will determine, up to the conjugacy, all the maximal subgroups of PSL(11, 2) by Aschbacher's theorem.

1. INTRODUCTION

The purpose of this paper is to prove the following main theorem:

Theorem (1.1): Let G = PSL(11, 2). If H is a maximal subgroup of G, then H isomorphic to one of the following subgroups:

1. A group $G_{(p)}$ or $G_{(9\text{-}\pi)}$, stabilizing a point or its dual, the stabilizer of a hyperplane. These are isomorphic to a group of form 2^{10}**.** SL(10, 2).
2. A group $G_{(l)}$ or $G_{(8\text{-}\pi)}$, stabilizing a line or its dual, the stabilizer of a 8-space. These are isomorphic to a group of form 2^{18}**.** (SL(2, 2) × SL(9, 2)).
3. A group $G_{(2\text{-}\pi)}$, or $G_{(7\text{-}\pi)}$, stabilizing a plane or its dual, the stabilizer of a 7-space. These are isomorphic to a group of form 2^{24}**.** (SL(3, 2) × SL(8, 2)).
4. A group $G_{(3\text{-}\pi)}$, or $G_{(6\text{-}\pi)}$, stabilizing a 3-space or its dual, the stabilizer of a 6-space. These are isomorphic to a group of form 2^{28}**.** (SL(4, 2) × SL(7, 2)).
5. A group $G_{(4\text{-}\pi)}$, or $G_{(5\text{-}\pi)}$, stabilizing a 4-space or its dual, the stabilizer of a 5-space. These are isomorphic to a group of form 2^{30}**.** (SL(5, 2) × SL(6, 2)).
6. A Singer cycle subgroup H = ΓL(1, 2^{11}).
7. PΓL(2, 23).
8. Mathieu group M_{24}.

Through this paper, ΓL(n, q) denote the group of all non-singular semi-linear transformation of a vector space $V_n(q)$ of dimension n over a field F_q with q is a prime power. *The general linear group* GL(n, q), consisting of the set of all invertible n×n matrices. In fact, GL(n, q) is a subgroup of ΓL(n, q) consisting of all non-singular linear transformations of $V_n(q)$. *The centre* Z of GL(n, q) is the set of all non-singular scalar matrices. The factor group

GL(n, q) / Z called *The projective general linear group* which is denoted by PGL(n, q). GL(n, q) has a normal subgroup SL(n, q), consisting of all matrices of determinant 1 called *the special linear group*. *The projective special linear group* PSL(n, q) is the quotient group SL(n, q) /(Z ∩ SL(n, q)). PSL(n, q) is simple, except for PSL(2, 2) and PSL(2, 3).

PG(n-1, q) will denote *the projective space* of dimension n-1 associated with $V_n(q)$. One, two and three- dimensional subspaces of $V_n(q)$ will be called *points*, *lines* and *planes* respectively. An (n-1)-dimensional subspace shall be called *a hyperplane*. An element T ∈ GL(n, q) is called *a transvection* if T satisfies rank ($T - I_n$) = 1 and $(T - I_n)^2 = 0$.

A split extension (*a semidirect product*) A**:**B is a group G with a normal subgroup A and a subgroup B such that G = AB and A∩B = 1. *A non-split extension* A**.**B is a group G with a normal subgroup A and G/A ≅ B, but with no subgroup B satisfying G = AB and A∩B = 1. A group G = A ∘ B is *a central product* of its subgroups A and B if G = AB and [A, B], the commutator of A and B = {1}, in this case A and B are normal subgroups of G and A∩B ≤ Z(G). If A∩B = {1}, then A ∘ B = AB.

G = PSL (11, 2) is a simple group of order 768105432118265670534631586896281600, thus $|G| = 2^{55}.3^6.5^2.7^3.11.17.23.31^2.73.89.127$ acting as a doubly transitive permutation group on the points of the projective space PG(10, 2).

2. ASCHBACHER'S THEOREM

In this section, we will give some definitions before starting a brief description of Aschbacher's theorem (2).

Definition (2.1) :

Let V be a vector space of dimensional n over a finite field q, a subgroup H of GL(n, q) is called *reducible* if it stabilizes a proper nontrivial subspace of V. If H is not reducible, then it is called *irreducible*. If H is irreducible for all field extension F of F_q, then H is *absolutely irreducible*. An irreducible subgroup H of GL(n, q) is called *imprimitive* if there are subspaces $V_1, V_2, \ldots, V_k$, $k \geq 2$, of V such that $V = V_1 \oplus \ldots \oplus V_k$ and H permutes the elements of the set $\{ V_1, V_2, \ldots, V_k\}$ among themselves. When H is not imprimitive then it is called *primitive*.

Definition (2.2):

A group G ≤ GL(n, q) is a *superfield group* of degree s if for some s divides n with s > 1, the group G may be embedded in $\Gamma L(n/s, q^s)$.

Definition (2.3) :

If the group G ≤ Gl(n, q) preserves a decomposition $V = V_1 \otimes V_2$ with $\dim(V_1) \neq \dim(V_2)$ then G is a *tensor product group.*

Suppose that $n = r^m$ for m > 1. If G ≤ Gl(n, q) preserves a decomposition $V = V_1 \otimes \ldots \otimes V_m$ with $\dim(V_i) = r$ for $1 \leq i \leq m$, then G is *tensor induced* group.

Definition (2.4):
A group $G \leq Gl(n, q)$ is *subfield group* if there exists a subfield $F_{q_o} \subset F_q$ such that G can be embedded in $GL(n, q_o) . Z$.

Definition (2.5):
A p-group G is called *special* if $Z(G)= G'$ and is called *extraspecial* if also $|Z(G)| = p$.

Definition (2.6) :
Let Z denote the group of scalar matrices of G. Then G is *almost simple modulo scalars* if there is a non-abelian simple group T such that $T \leq G/Z \leq Aut(T)$, the automorphism group of T.

A classification of the maximal subgroups of GL(n, q) by Aschbacher's theorem (2), which may be briefly summarized as follows:

Result (2.7) (Aschbacher's theorem):- (2).
Let H be a subgroup of GL(n, q), $q = p^e$ with the center Z and let V be the underlying n-dimensional vector space over a field q. If H is a maximal subgroup of GL(n, q), then one of the following holds:

C_1:- H is a reducible group.
C_2:- H is an imprimitive group.
C_3:- H is a superfield group.
C_4:- H is a tensor product group.
C_5:- H is a subfield group.
C_6:- H normalizes an irreducible extraspecial or symplectic-type group.
C_7:- H is a tensor induced group.
C_8:- H normalizes a classical group in its natural representation.
C_9:- H is absolutely irreducible and $H /(H \cap Z)$is almost simple.

To prove theorem (1.1) by using Aschbacher's theorem (Result (2.7)), first, we will determine the maximal subgroups in the classes $C_1 – C_8$ of Aschbacher's theorem (Result (2.7)):

3. CLASSES C_1–C_8 OF ASCHBACHER'S THEOREM (RESULT (2.7))

3.1 The subgroups of C_1:

Let H be a reducible subgroup of G and W an invariant subspace of H. If we let d = dim (W), then$1 \leq d \leq 11$. Let $G_d = G_{(W)}$ denote the subgroup of G containing all elements fixing Was a whole and $H \subseteq G_{(W)}$. with a suitable choice of a basis, $G_{(W)}$ consists of all matrices of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A and C are d × d and (11-d) ×(11-d) non-singular matrices of determinant 1, where B is an arbitrary d×(11-d) matrix. G_d is isomorphic to a group of the form $2^{d(11-d)}$ (SL(d, 2)) × (SL(11-d, 2)).

which give us the following reducible maximal subgroups of G:

1. A group $G_{(p)}$ or $G_{(9-\pi)}$, stabilizing a point or its dual, the stabilizer of a hyperplane. These are isomorphic to a group of form 2^{10}**.** SL(10, 2).

2. A group $G_{(l)}$ or $G_{(8-\pi)}$, stabilizing a line or its dual, the stabilizer of a 8-space. These are isomorphic to a group of form 2^{18}. (SL(2, 2) × SL(9, 2)).
3. A group $G_{(2-\pi)}$, or $G_{(7-\pi)}$, stabilizing a plane or its dual, the stabilizer of a 7-space. These are isomorphic to a group of form 2^{24}. (SL(3, 2) × SL(8, 2)).
4. A group $G_{(3-\pi)}$, or $G_{(6-\pi)}$, stabilizing a 3-space or its dual, the stabilizer of a 6-space. These are isomorphic to a group of form 2^{28}. (SL(4, 2) × SL(7, 2)).
5. A group $G_{(4-\pi)}$, or $G_{(5-\pi)}$, stabilizing a 4-space or its dual, the stabilizer of a 5-space. These are isomorphic to a group of form 2^{30}. (SL(5, 2) × SL(6, 2)).

Which prove the points (1), (2), (3), (4) and (5) of the main theorem (1.1).

3.2 The maximal subgroups of C_2:

If H is imprimitive, then H preserves a decomposition of V as a direct sum $V = V_1 \oplus_{\dots} \oplus V_t$, t >1, into subspaces of V, each of dimension m = n/t, which are permuted transitively by H, thus C_2 are isomorphic to GL(m, q)**:**S_t.

Consequently, there are no C_2 groups in PSL(11, 2) since 11 is a prime number.

Note: if q > 2, then there exist an imprimitive group $G_{(\Delta)}$ of order n! $(q-1)^{n-1}$ preserving a n-simplex points of PG(n-1, q) with coordinates in F_q and $G_{(\Delta)}$ interchanges them. Consequently, there is no $G_{(\Delta)}$ subgroup in PSL(11, 2), since q = 2 is not greater than 2.

3.3 The maximal subgroups of C_3:

If H is (superfield group) a semilinear groups over extension fields of GF(q) of prime degree, then H acts on G as a group of semilinear automorphism of a (n/k)-dimensional space over the extension field $GF(q^k)$, so H embeds in $\Gamma L(n/k, q^k)$, for some prime number k dividing n.

Consequently, there are no C_3 groups in PSL(11, 2) since 11 is a prime number.

Definition (3.3.1) : *A Singer cycle* of GL(n, q) is an element of order q^n-1.

Result (3.3.2): (14) , (20) and (31).

If n is a prime number, then there exist a Singer cycles group H = $\Gamma L(1, q^n)$ of order $d^{-1}(q^n-1)/(q-1)$, where d = gcd(n, q-1) and H is irreducible maximal subgroup of PSL(n, q) which it is the normalizer of the (cyclic) multiplicative group for $GF(q^n)$.

Consequently, there is a Singer cycle subgroup H = $\Gamma L(1, 2^{11})$ in PSL(11, 2), since 11 is a prime number which prove the point (6) of the main theorem (1.1).

3.4 The maximal subgroups of C_4:

If H is a tensor product group, then H preserves a decomposition of V as a tensor product $V_1 \otimes V_2$, where $\dim(V_1) \neq \dim(V_2)$ of spaces of dimensions k, m > 1 over GF(q), and so H stabilize the tensor product decomposition $F^k \otimes F^m$, where n = km, k ≠ m. Thus, H is a subgroup of the central product of GL(k, q) ∘ GL(m, q).

Consequently, there is no tensor product group in PSL(11, 2), since 11 can not be analysis to two different numbers.

3.5 The maximal subgroups of C_5:

If H is a subfield group, then H is the linear groups over subfields of GF(q) of prime index. Thus H can be embedded in $GL(n, p^f)$.Z where e/f is prime number and $q = p^e$.

Consequently, there are no C_5 groups in PSL(11, 2), since 2 is a prime number.

3.6 The maximal subgroups of C_6:

For the dimension $n = r^m$, if r is prime number divides q-1, then $H = r^{2m} : Sp(2m, r)$ is an extraspecial r-group of order r^{2m+1}, or if r = 2 and 4 divides q-1, then $H = 2^{2m} . O^{\in}(2m, 2)$ normalizes a 2-group of symplectic type of order 2^{2m+2}.

Consequently, there are no C_6 groups in PSL(11, 2), since n = 11 is not prime power.

3.7 The maximal subgroups of C_7:

If H is a tensor-induced, then H preserves a decomposition of V as $V_1 \otimes V_2 \otimes \dots \otimes V_m$ where V_i are isomorphic and each V_i has dimension $r > 1$, $n = \dim V = r^m$, and the set of V_i is permuted by H, so H stabilize the tensor product decomposition $F^r \otimes F^r \otimes \dots \otimes F^r$, where $F = F_q$. Thus $H/Z \leq PGL(r, q) : S_m$.

Consequently, there is a tensor-induced group in PSL(11, 2), since n = 11 is not prime power.

3.8 The maximal subgroups of C_8:

If H normalizes a classical group in its natural representation, then H lies between a classical group and its normalizer in GL(n, q), so H preserves a classical form up to scalar multiplication. Thus H is a normalizer of such a subgroup $PSL(n, q')$, $PSp(n, q')$, $P\Omega(n, q')$ or $PSU(n, q')$ for various q' dividing q.

Consequently, there are no C_8 groups in PSL(11, 2), since 2 is not a square, and is odd number.

Finally, we will determine the maximal subgroups in class C_9 of Aschbacher's theorem {Result (2.7)}:

4. The maximal subgroups of C_9:

If H is absolutely irreducible and $H/(H \cap Z)$is almost simple, then H is the normalizer of absolutely irreducible normal subgroup M of H which is non-abelian and simple group.

To find the maximal subgroups of C_9, we will determine the maximal primitive subgroups H of G which have the property that a minimal normal subgroup M of H is non abelian group.

The following corollary will play an important role in proving the main result of this section {theorem (4.2)}

Corollary (4.1): If M is a non abelian simple group of a primitive subgroup H of G, then M is isomorphic to one of the following groups:

a) PSL(2, 23).

b) Mathieu groups, M_{23} or M_{24}.

Proof: let H be a primitive subgroup of G with a minimal normal subgroup M of H is not abelian. So, we will discuss the possibilities of a minimal normal subgroup M of H according to:

(I) M contains transvections. {(section (4.1)}

(II) M does not contain any transvection. {(section (4.2)}

(III) M is doubly transitive. {(section (4.3)}.

4.1 Primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian is generated by transvections:

To find the primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian is generated by transvections, we will use the following result of Mclaughlin (25):

Result (4.1.1) (Mclaughlin Theorem) (25):

Let H be a proper irreducible subgroup of SL(n, 2) generated by transvections. Then n > 3 and H is Sp(n, 2), $O^{\in}(n, 2)$, S_{n+1} or S_{n+2}.

In the following, we will discuss the different possibilities of Result (4.1.1), which will give us the following main result of section (4.1):

Corollary (4.1.2): There is no proper irreducible subgroup H of SL(11, 2) generated by transvections.

Proof:

From Mclaughlin Theorem {Result (4.1.1)}, M is isomorphic to one of the following groups: symplectic group, orthogonal groups O(11, 2), symmetric groups S_{12} or S_{13}.

1. There is no symplectic groups since n is odd number.
2. From the character table of the orthogonal group O(11, 2) by GAP:

```
gap> g:=GO(11,2);
GO(0,11,2)
gap> c:=CharacterTable("g");
CharacterTable( "4.2^4.S5" )
gap> k:=CharacterTable(c, 2);
BrauerTable( "4.2^4.S5", 2 )
gap> CharacterDegrees(k);
[ [ 1, 1 ], [ 4, 2 ] ]
```

And non of them of degree 11. Thus, if O(11, 2) $\subset$ G, then it must be reducible.

3. From the character table of S_{12}, G contain no class of subgroups isomorphic to S_{12}.

[[1, 1], [10, 1], [32, 1], [44, 1], [100, 1], [164, 1], [288, 1], [320, 1], [416, 1], [570, 1], [1046, 1], [1408, 1], [1792, 1], [2368, 1], [5632, 1]]

(gap> CharacterDegrees(CharacterTable("S12")mod 2);)

And non of them of degree 11. Thus $S_{12} \not\subset$ G.

4. From the character table of S_{13}, G contain no class of subgroups isomorphic to S_{13}.

[[1, 1], [12, 1], [64, 2], [208, 1], [288, 1], [364, 2], [560, 1], [570, 1], [1572, 1], [1728, 1], [2208, 1], [2510, 1], [2848, 1], [3200, 1], [8008, 1], [8448, 1]]

(gap> CharacterDegrees(CharacterTable("S13")mod 2);)

And non of them of degree 11. Thus $S_{13} \not\subset$ G.

4.2 Primitive subgroups H of G which have the property that a minimal normal subgroup of H is not abelian and does not contain transvections:

In this section, we will consider a minimal normal subgroup M of H is not abelian and does not contain any transvections.

The following corollary is the main result of section (4.2):

Corollary (4.2.1):
If Y be a non - abelian simple subgroup of G which does not contain any transvection. Then Y is isomorphic to PSL(2, 23).

Proof:

We will prove Corollary (4.2.1) by series of Lemmas (4.2.2) through Lemmas (4.2.8) and Result (4.2.3).

Lemma (4.2.2):
Let Y is a primitive subgroup of G such that Y does not contain any transvection. If S(2) be a 2-Sylow subgroup of Y, then S(2) contains no elementary abelian subgroup of order 8.

Proof:

A 2-Sylow subgroup of G can be represented by the set of all matrices of the form:

$$\begin{bmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\ & 1 & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ & & 1 & x_{20} & x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} \\ & & & 1 & x_{28} & x_{29} & x_{30} & x_{31} & x_{32} & x_{33} & x_{34} \\ & & & & 1 & x_{35} & x_{36} & x_{37} & x_{38} & x_{39} & x_{40} \\ & & & & & 1 & x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ & & & & & & 1 & x_{46} & x_{47} & x_{48} & x_{49} \\ & & & & & & & 1 & x_{50} & x_{51} & x_{52} \\ & & & & & & & & 1 & x_{53} & x_{54} \\ & & & & & & & & & 1 & x_{55} \\ & & & & & & & & & & 1 \end{bmatrix}$$

Where all entries are in F_2.

Let Y is a primitive subgroup of G such that Y does not contain any transvection. If S(2) be a 2-Sylow subgroup of Y, then inside S(2), there exist only two elementary abelian subgroups of the form:-

$$A = \left\{ \begin{bmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\ & 1 & . & . & . & . & . & . & . & . & . \\ & & 1 & . & . & . & . & . & . & . & . \\ & & & 1 & . & . & . & . & . & . & . \\ & & & & 1 & . & . & . & . & . & . \\ & & & & & 1 & . & . & . & . & . \\ & & & & & & 1 & . & . & . & . \\ & & & & & & & 1 & . & . & . \\ & & & & & & & & 1 & . & . \\ & & & & & & & & & 1 & . \\ & & & & & & & & & & 1 \end{bmatrix} \right\} \quad \text{and} \quad B = \left\{ \begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . & x_1 \\ & 1 & . & . & . & . & . & . & . & . & x_2 \\ & & 1 & . & . & . & . & . & . & . & x_3 \\ & & & 1 & . & . & . & . & . & . & x_4 \\ & & & & 1 & . & . & . & . & . & x_5 \\ & & & & & 1 & . & . & . & . & x_6 \\ & & & & & & 1 & . & . & . & x_7 \\ & & & & & & & 1 & . & . & x_8 \\ & & & & & & & & 1 & . & x_9 \\ & & & & & & & & & 1 & x_{10} \\ & & & & & & & & & & 1 \end{bmatrix} \right\}$$

where the orders of A and B are equal to 2^{10}

A corresponds to transvections:

$$I + \begin{bmatrix} 1 \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \end{bmatrix} \begin{bmatrix} . & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{bmatrix}$$

And B corresponds to transvections :

$$I+\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ . \end{bmatrix}[. \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; 1]$$

Since S(2) does not contain any transvections, then both A and B must be the identity element. Then S(2) contains no elementary abelian subgroup of order 8.

Result (4.2.3): (1)
Let Y be a simple group. Assume that the 2-Sylow subgroup of Y contains no elementary abelian subgroup of order 8. Then Y is isomorphic to one of the following groups: A_7, PSL(2, q), PSL(3, q), PSU(3, q) with q odd or PSU(3, 4).

We will proceed to determine which of these groups satisfy the conditions of Corollary (4.2.1).

Lemma (4.2.4): $A_7 \not\subset G$

Proof:

Since the irreducible 2-modular characters for A_7 by GAP are:

[[1, 1], [4, 2], [6, 1], [14, 1], [20, 1]]

(gap > CharacterDegrees (CharacterTable ("A7") mod 2));

And non of them of degree 11.

Lemma (4.2.5): If PSL(2, q) $\subset$ G, q odd, then q = 23.

Proof:

PSL(2, q) has no projective representation in G of degree < (1/2)(q-1) {(22) and (29)} and (1/2)(q-1) > 11 for all odd q > 23. Hence we need only to consider the cases when q $\leq$ 23.

a. PSL(2, 3) is not simple.

b. PSL(2, 5) $\cong$ PSL(2, 2^2),

The irreducible 2-modular characters for PSL(3, 5) by GAP are:

[[1, 1], [2, 2], [4, 1]],

(gap > CharacterDegrees (CharacterTable (" L2(5) ") mod 2));

But non of them of degree 11. Therefore if PSL(2, 5) $\subset$ G, then it is reducible.

c. PSL(2, 7) $\cong$ PSL(3, 2),

The irreducible 2-modular characters for PSL(2,7) by GAP are:

[[1, 1], [3, 2], [8, 1]],

(gap > CharacterDegrees (CharacterTable (" L2(7) ") mod 2));

But non of them of degree 11. Therefore if PSL(2, 7) $\subset$ G, then it is reducible.

d. For PSL(2, 3^2) $\cong$ A_6:

The irreducible 2-modular characters for PSL(2, 3^2) by GAP are:

[[1, 1], [4, 2], [8, 2]].

(gap > CharacterDegrees (CharacterTable (" L2(9) ") mod 2));

But non of them of degree 11. Therefore if PSL(2, 3^2) $\subset$ G, then it is reducible.

e. For PSL(2, 11):

The irreducible 2-modular characters for PSL(2, 11) by GAP are:

[[1, 1], [5, 2], [10, 1], [12, 2]].

(gap > CharacterDegrees (CharacterTable ("L2(11) ") mod 2));

But non of them of degree 11. Therefore if PSL(2, 11) $\subset$ G, then it is reducible.

f. For PSL(2, 13):

The irreducible 2-modular characters for PSL(2, 13) by GAP are:

[[1, 1], [6, 2], [12, 3], [14, 1]].

(gap > CharacterDegrees (CharacterTable (" L2(13) ") mod 2));

But non of them of degree 11. Therefore if PSL(2, 13) $\subset$ G, then it is reducible.

g. For PSL(2, 17):

The irreducible 2-modular characters for PSL(2, 13) by GAP are:

[[1, 1], [8, 2], [16, 4]],

(gap > CharacterDegrees (CharacterTable (" L2(17) ") mod 2));

But non of them of degree 11. Therefore if PSL(2, 17) $\subset$ G, then it is reducible.

h. For PSL(2, 19):

The irreducible 2-modular characters for PSL(2, 19) by GAP are:

[[1, 1], [9, 2], [18, 2], [20, 4]],

(gap > CharacterDegrees (CharacterTable (" L2(19) ") mod 2)   );

But non of them of degree 11. Therefore if PSL(2, 19) $\subset$ G, then it is reducible.

i. For PSL(2, 23):

The irreducible 2-modular characters for PSL(2, 23) by GAP are:

[[1, 1], [11, 2], [22, 1], [24, 5]]

gap> CharacterDegrees(CharacterTable("PSL(2,23)")mod 2);

Hence, there are two classes of degree 11. Therefore PSL(2, 23) $\subset$ G

Lemma (4.2.6): PSL(3, q) $\not\subset$ G, for all q.

Proof:

PSL(3, q) has no projective representation in G of degree $<q^{n-1}-1 = q^2-1$ {(22) and (29)}, and it is clear that $q^2-1 > 11$ for all $q \geq 4$. Thus, we need to test PSL(3, 2) and PSL(3, 3) as primitive subgroups of G ?

- PSL(3, 2) $\not\subset$ G, [see lemma (4.2.5)]
- PSL(3, 3) $\not\subset$ G, since the irreducible 2-modular characters for PSL(3, 3) by GAP are:

[1, 1], [12, 1], [16, 4], [26, 1]],

(gap > CharacterDegrees (CharacterTable (" PSL(3, 3) ") mod 2));

Hence, non of these is of degree 11, therefore if PSL(3, 3) $\subset$ G, then it is reducible.

Lemma (4.2.7): PSU(3, q) $\not\subset$ G, for all q.

Proof:

PSU(3, q) has no projective representation in G of degree < q(q-1), (29), and it is clear that q(q-1) > 11 for all $q \geq 4$. Thus, we need to test PSU(3, 2) and PSU(3, 3) are primitive subgroups of G ?

- PSU(3, 2) is not simple.

- PSU(3, 3) ⊄ G, since the irreducible 2-modular characters for PSU(2, 9) by GAP are: [[1, 1], [6, 1], [14, 1], [32, 2]]

(gap> CharacterDegrees(CharacterTable("U3(3)")mod 2)).

and non of these of degree 11.

Lemma (4.2.8): PSU(3, 4) ⊄ G.

Proof:

PSU(3, 4) does not satisfy the conditions of this section, since PSU(3, 4) is not simple.

4.3 Primitive subgroups H of G which have the property that a minimal normal subgroup of H which is not abelian is doubly transitive group:

In this section, we will consider a minimal normal subgroup M of H is not abelian and is doubly transitive group:

The following Corollary is the main result of this section:

Corollary (4.3.1): If M is a non abelian simple group of doubly transitive group H, then M is isomorphic to one of the following groups:

a) PSL(2, 23).

b) Mathieu groups, M_{23} or $M_{24.}$

Proof:

Since every doubly transitive group is a primitive group (3), then we will use the classification of doubly transitive groups { (13) and (26) }. And we will prove Corollary (4.3.1) by series of Lemmas (4.3.3) through Lemmas (4.3.15) and Result (4.3.2).

Result (4.3.2): { (13) and (26)}.

If Y be a doubly transitive group, then Y has a simple normal subgroup M^*, and $M^* \subseteq Y \subseteq Aut(M^*)$, where M^* as follows:

1. A_n, $n \geq 5$;
2. PSL(d, q), $d \geq 2$, where $(d, q) \neq (2, 2), (2, 3)$;
3. PSU(3, q), $q > 2$;
4. the Suzuki group Sz(q), $q = 2^{2m+1}$ and $m > 0$;
5. the Ree group Re(q), $q = 3^{2m+1}$ and $m > 0$;
6. Sp(2n, 2), $n \geq 3$;
7. PSL(2, 11);
8. Mathieu groups M_n, n = 11, 12, 22, 23, 24.
9. HS (Higman-Sims group);
10. CO_3 (Conway's smallest group).

In the following, we will discuss the different possibilities of Result (4.3.2);

Lemma (4.3.3): $A_n \not\subset G$, for all $n \geq 5$.

Proof:

From (30) , A_n for all $n > 8$, has a unique faithful 2-modular representation of least degree, this degree being (n-1) if n is odd and (n-2) if n is even, so, the 2-modular representation of least degree is greater than 11 for all $n \geq 14$. Thus $A_n \not\subset G$ for any $n \geq 14$.

$A_5 \not\subset G$: since the irreducible 2-modular characters for A_5 by GAP are:

[[1, 1], [2, 2], [4, 1]]

(gap > CharacterDegrees (CharacterTable ("A5") mod 2));

$A_6 \not\subset$ G: since the irreducible 2-modular characters for A_6 by GAP are:
[[1, 1], [4, 2], [8, 2]]
(gap > CharacterDegrees (CharacterTable ("A6") mod 2));
$A_7 \not\subset$ G: since the irreducible 2-modular characters for A_7 by GAP are:
[[1, 1], [4, 2], [6, 1], [14, 1], [20, 1]]
(gap > CharacterDegrees (CharacterTable ("A7") mod 2));
$A_8 \not\subset$ G: since the irreducible 2-modular characters for A_8 by GAP are:
[[1, 1], [4, 2], [6, 1], [14, 1], [20, 2], [64, 1]]
(gap > CharacterDegrees (CharacterTable ("A8") mod 2));
$A_9 \not\subset$ G: since the irreducible 2-modular characters for A_9 by GAP are:
[[1, 1], [8, 3], [20, 2], [26, 1], [48, 1], [78, 1], [160, 1]]
(gap > CharacterDegrees (CharacterTable ("A9") mod 2));
$A_{10} \not\subset$ G: since the irreducible 2-modular characters for A_{10} by GAP are:
[[1, 1], [8, 1], [16, 1], [26, 1], [48, 1], [64, 2], [160, 1], [198, 1], [200, 1], [384, 2]]
(gap > CharacterDegrees (CharacterTable ("A10") mod 2)).
$A_{11} \not\subset$G: since the irreducible 2-modular characters for A_{11} by GAP are:
[[1, 1], [10, 1], [16, 2], [44, 1], [100, 1], [144, 1], [164, 1], [186, 1], [198, 1], [416, 1], [584, 2], [848, 1]],
(gap > CharacterDegrees (CharacterTable ("A11") mod 2));
$A_{12} \not\subset$ G: since the irreducible 2-modular characters for A_{12} by GAP are:
[1, 1], [10, 1], [16, 2], [44, 1], [100, 1], [144, 2], [164, 1], [320, 1], [416, 1], [570, 1], [1046, 1], [1184, 2], [1408, 1], [1792, 1], [5632,1].
(gap > CharacterDegrees (CharacterTable ("A12") mod 2));
$A_{13} \not\subset$ G: since the irreducible 2-modular characters for A_{12} by GAP are:
[[1, 1], [12, 1], [32, 2], [64, 1], [144, 2], [208, 1], [364, 2], [560, 1], [570, 1], [1572, 1], [1728, 1], [2208, 1], [2510, 1], [2848, 1], [3200, 1], [4224, 2], [8008, 1]]
(gap > CharacterDegrees (CharacterTable ("A13") mod 2));

Lemma (4.3.4): If PSL(2, q) $\subset$ G, then q = 23

Proof:

We have two cases:

Case (1). q is even:

PSL(2, q) has no projective representation in G of degree < (1/d)(q-1), d = g.c.d(2, q-1) {(22) and (29)}, and (q-1) > 11 for all even q $\geq$ 16. Also,

- PSL(2, 2) not simple.
- PSL(2, 4) $\not\subset$ G, since the irreducible 2-modular characters for PSL(2, 4) by GAP are:

[[1, 1], [2, 2], [4, 1]],
(gap > CharacterDegrees (CharacterTable (" L2(4) ") mod 2));
and non of these of degree 11.

- PSL(2, 8) $\not\subset$ G, since the irreducible 2-modular characters for PSL(2, 8) by GAP are:

[[1, 1], [2, 3], [4, 3] , [8, 1]],

(gap > CharacterDegrees (CharacterTable (" L2(4) ") mod 2));
and non of these of degree 11.
Thus, PSL(2, q) $\not\subset$ G for all q is even.
Case (2). q is odd:
If PSL(2, q) $\subset$ G, q is odd, then q = 23. [see Lemma (4.2.5)]

Lemma (4.3.5): PSL(n, 2) $\not\subset$ G for all n.

Proof:

PSL(n, 2) has no projective representation in G of degree < q^{n-1}-1 = 2^{n-1}-1 { (22) and (29)}, and it is clear that 2^{n-1}-1 > 11 for all n > 4. Thus, we need to test PSL(2, 2), PSL(3, 2) and PSL(4, 2) are primitive subgroups of G ?

- PSL(2, 2) is not simple.
- PSL(3, 2) $\not\subset$ G. Since PSL(3, 2) ≅ PSL(2, 7), and PSL(2, 7) $\not\subset$ G. [see Lemma(4.2.5)]
- PSL(4, 2) $\not\subset$ G. Since PSL(4, 2) ≅ A_8, and $A_8 \not\subset$ G [see Lemma(4.2.5)]

Lemma (4.3.6): If PSL(n, q) $\subset$ G, then n = 2 and q = 23

Proof:

PSL(n, q) has no projective representation in G of degree < (q^{n-1}-1) {(22) and (29) }, which > 11 for all for all q $\geq$ 3 and n $\geq$ 4. Thus, we need to test PSL(2, q), PSL(3, q) and PSL(n, 2) as primitive subgroups of G ?

- If PSL(2, q) $\subset$ G, then q = 23 [see lemma (4.3.4)].
- PSL(3, q) $\not\subset$ G for all q [see Lemma (4.2.6)].
- PSL(n, 2) $\not\subset$ G for all n [see Lemma (4.3.5)].

Lemma (4.3.7): PSU(2, q) $\not\subset$ G, for all q.

Proof:

PSU(2, q) $\subseteq$ PGL(2, q). But PGL(2, q) has no projective representation in G of degree < (q-1), provided q $\neq$ 9 (29), which > 11 for all q > 13.

Thus, we need to test PSU(2, 2) , PSU(2, 3), PSU(2, 4), PSU(2, 5), PSU(2, 7), PSU(2, 9), PSU(2, 11) and PSU(2, 13) are primitive subgroups of G ?

- PSU(2, 2) is not simple.
- PSU(2, 3) is not simple.
- PSU(2, 4) $\not\subset$ G, since the irreducible 2-modular characters for PSU(2, 4) by GAP are: [[1, 1], [2, 2], [4, 1]]

(gap> CharacterDegrees(CharacterTable("U2(4)")mod 2))
and there is non of degree 11.

- PSU(2, 5) $\not\subset$ G, since the irreducible 2-modular characters for PSU(2, 5) by GAP are: [[1, 1], [2, 2], [4, 1]]

(gap> CharacterDegrees(CharacterTable("U2(5)")mod 2)).
and there is non of degree 11.

- PSU(2, 7) $\not\subset$ G, since the irreducible 2-modular characters for PSU(2, 7) by GAP are: [[1, 1], [3, 2], [8, 1]]

(gap> CharacterDegrees(CharacterTable("U2(7)")mod 2)).
and there is non of degree 11.

- PSU(2, 9) $\not\subset$ G, since the irreducible 2-modular characters for PSU(2, 9) by GAP are: [[1, 1], [4, 2], [8, 2]]

  (gap> CharacterDegrees(CharacterTable("U2(9)")mod 2)).

  and there is non of degree 11.
- PSU(2, 11) $\not\subset$ G, since the irreducible 2-modular characters for PSU(2, 11) by GAP are: [[1, 1], [5, 2], [10, 1], [12, 2]]

  (gap> CharacterDegrees(CharacterTable("U2(11)")mod 2)).

  and there is non of degree 11.
- PSU(2, 13) $\not\subset$ G, since the irreducible 2-modular characters for PSU(2, 13) by GAP are: [[1, 1], [6, 2], [12, 3], [14, 1]]

  (gap> CharacterDegrees(CharacterTable("U2(13)")mod 2)).

  and there is non of degree 11.

Lemma (4.3.8): PSU(n, 2) $\not\subset$ G, for all n.

Proof:

PSU(n, q), n $\geq$ 3, has no projective representation in G of degree < $q(q^{n-1}-1)/(q+1)$ if n is odd, and PSU(n, q) , n $\geq$3, has no projective representation in G of degree < $(q^{n}-1)/(q+1)$ if n is even. { (22) and (29)}, Thus the minimal projective degree for PSU(n, 2) is > 11 for all n $\geq$ 6.

Thus, we need to test PSU(2, 2) , PSU(3, 2), PSU(4, 2) and PSU(5, 2) are primitive subgroups of G ?

- PSU(2, 2^2) is not simple.
- PSU(3, 2^2) is not simple.
- PSU(4, 2) $\not\subset$ G. Since the irreducible 2-modular characters for PSU(4, 2) by GAP are: [[1, 1], [4, 2], [6, 1], [14, 1], [20, 2], [64, 1]]

  (gap> CharacterDegrees(CharacterTable("U4(2)") mod 2)).

  and non of these of degree 11.
- PSU(5, 2) $\not\subset$ G, since the irreducible 2-modular characters for PSU(5, 2) by GAP are: [[1, 1], [5, 2], [10, 2], [24, 1], [40, 4], [74, 1], [160, 2], [280, 2], [1024, 1]]

  (gap> CharacterDegrees(CharacterTable("U5(2)") mod 2)).

Lemma (4.3.9): PSU(n, q) $\not\subset$ G.

Proof:

PSU(n, q), n $\geq$ 3, has no projective representation in G of degree < $q(q^{n-1}-1)/(q+1)$ if n is odd, and PSU(n, q) , n $\geq$3, has no projective representation in G of degree < $(q^{n}-1)/(q+1)$ if n is even. { (22) and (29)}, Thus the minimal projective degree is > 11 for all n > 3 and q $\geq$ 3.

Thus, we need to test PSU(n, 2), PSU(2, q) and PSU(3, q) are primitive subgroups of G ?

- PSU(n, 2) $\not\subset$ G, [see Lemma (4.3.8)].
- PSU(2, q) $\not\subset$ G, [see Lemma (4.3.7)].
- PSU(3, q) $\not\subset$ G, [see lemma (4.2.7)].

Lemma(4.3.10): Sz(q) $\not\subset$ G, q = 2^{2m+1} and m > 0.

Poof:

The irreducible 2-modular characters for Suzuki groups by GAP are:

[[1, 1], [4, 3], [16, 3], [64, 1]]

(gap > CharacterDegrees (CharacterTable (" Sz(8) ") mod 2));

and non of these of degree 11, thus Sz(q) $\not\subset$ G.

Lemma (4.3.11): Re(q) $\not\subset$ G, q = 3^{2m+1}.

Proof:

The irreducible 2-modular characters for Ree group Re(q) by GAP are: [[1, 1], [702, 1], [741, 2], [2184, 2], [13832, 6], [16796, 1], [18278, 1], [19684, 6], [26936, 3]]

(gap> CharacterDegrees (CharacterTable (" R(27) ") mod 2));

and non of these of degree 11, thus Re(q) $\not\subset$ G.

Lemma (4.3.12): PSp(2n, 2) $\not\subset$ G for all n $\geq$3.

Proof:

From {(22) and (29)}, PSp(2n, q), n $\geq$ 2 has no projective representation in G of degree < $(1/2)q^{n-1}(q^{n-1} – 1)(q-1)$ if q is even. And since q = 2, then $(1/2)q^{n-1}(q^{n-1} – 1)(q-1)$ > 11 for all n $\geq$ 4. Thus, we need to test PSp(6, 2) is a primitive subgroups of G ?

The irreducible 2-modular characters for PSp(6, 2) by GAP are: [1, 1], [6, 1], [8, 1], [14, 1], [48, 1], [64, 1], [112, 1], [512, 1]]

(gap> CharacterDegrees(CharacterTable(“S6(2)”)mod 2);

and non of these of degree 11, thus PSp(6, 2) $\not\subset$ G

Lemma (4.3.13): if the Mathieu groups M_n, n = 11, 12, 22, 23, 24 are primitive subgroups of G, then n = 23 or 24.

Proof:

- M_{11} $\not\subset$ G, since the irreducible 2-modular characters for Mathieu group M_{11} by GAP are:

[[1, 1], [10, 1], [16, 2], [44, 1]],

(gap > CharacterDegrees (CharacterTable (“ M11 “) mod 2));

- M_{12} $\not\subset$ G, since the irreducible 2-modular characters for Mathieu group M_{12} by GAP are:

[[1, 1], [10, 1], [16, 2], [44, 1], [144, 1]],

(gap > CharacterDegrees (CharacterTable (“ M12 “) mod 2));

- M_{22} $\not\subset$ G, since the irreducible 2-modular characters for Mathieu group M_{22} by GAP are:

[[1, 1], [10, 2], [34, 1], [70, 2], [98, 1]],

(gap > CharacterDegrees (CharacterTable (“ M22 “) mod 2)).

- M_{23} $\subset$ G, since the irreducible 2-modular characters for Mathieu group M_{23} by GAP are:

[[1, 1], [11, 2], [44, 2], [120, 1], [220, 2], [252, 1], [896, 2]]

gap> CharacterDegrees(CharacterTable(“M23”)mod 2);

- M_{24} $\subset$ G, since the irreducible 2-modular characters for Mathieu group M_{24} by GAP are:

[[1, 1], [11, 2], [44, 2], [120, 1], [220, 2], [252, 1], [320, 2], [1242, 1], [1792, 1]].

Gap> CharacterDegrees(CharacterTable("M24")mod 2);

Which prove the point (b) of Corollary (4.3.1).

Lemma (4.3.14): HS (Higman-Sims group) $\not\subset$ G;

Proof:

The minimal degrees of faithful representations of the Higman-Sims group over F_2 is 20, which is greater than 11, (Jansen, 2005).

Lemma (4.3.15): CO_3 (Conway's smallest group) $\not\subset$ G;

Proof:

The minimal degrees of faithful representations of the CO_3 over F_2 is 22, which is greater than 11 (Jansen, 2005).

Now, we will determine the maximal primitive group of C_9 :

Theorem (4.2): If H is a maximal primitive subgroup of G which has the property that a minimal normal subgroup M of H is not abelian group, then H is isomorphic to one of the following subgroups of G:

(1) PΓL (2, 23).

(2) Mathieu group M_{24}.

Proof:

We will prove this theorem by finding the normalizers of the groups of corollary (4.1) and determine which of them are maximal:

- The normalizer of PSL(2, 23) is PΓL(2, 23) { (16) , (17), (33) and (34)}. Thus PΓL(2, 23) is a maximal primitive subgroup of G.
- The normalizer of the Mathieu group M_{23} is the group M_{23} and the normalizer of the Mathieu group M_{24} is the group M_{24}, but M_{23} is a subgroup of M_{24} {(33) and (34)}. Thus M_{24} is a maximal primitive subgroup of G.

Which prove the points (7) and (8) of theorem (1.1), and this complete the proof of theorem (1.1).

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الزمر الجزئيه العظمى للزمره PSL(11, 2)

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ملخص

فى هذا البحث أوجدنا جميع الزمر الجزئيه العظمى للزمره الخطية PSL(11, 2) وذلك باستخدام نظريه أشبكا (2) لتعين الزمر الجزئيه العظمى للزمر الخطية ودونت النتيجه التى حصلنا عليها فى نظرية (1.1).