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# Inverse Linear Goal Programming Problem

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**ABSTRACT**

This paper considers the inverse linear goal programming problem of multi-objective function in case the change in coefficient of the objective function.
Let $S$ denote the set of feasible solutions points of linear goal programming problem of a multi-objective function, and let $d^{+o}, d^{-o}$ be the positive and negative deviation variables of the maximum and minimum goals respectively, $w^{+}, w^{-}$ be a specified cost vector, $x^{o}$ be given feasible solution vector, and $d^{+o}, d^{-o}$ be given tow vectors denoted the feasible positive deviation and the feasible negative deviation points of the max or min goals, respectively.

The inverse linear goal programming problem of multi-objective function is as follows:
Consider the change of the cost vectors $w^{+}, w^{-}$ as less as possible such that the vectors feasible solution $x^{o}, d^{+o}, d^{-0}$ becomes an optimal solution of LGP of multi-objective function under the new cost vectors $\lambda^{+}, \lambda^{-}$ and $\| |\lambda^{+} - w^{+}| + |\lambda^{-} - w^{-}| \|_{p}$ is minimal, where $\|.\|_{p}$ is some selected $L_{p}$-norm.

In this paper, we consider the inverse version ILGP of LGMP. under the $L_1$-norm where the objective is to minimize $\sum_{i\in I} |\lambda^{+} - w^{+}| d^{+} + |\lambda^{-} - w^{-}| d^{-}$, with $I$ denoting the index set of variables $x_i$. We show that the inverse version of the considered under $L_1$-norm reduces to solving a problem for the same kind; that is, an inverse multi-objective assignment problem reduces to an assignment problem.

## 1. INTRODUCTION

An inverse linear goal programming problem of multi-objective function is defined as follows:

Let $S$ denote the set of feasible solutions points of linear goal programming problem LGP of a multi-objective function, and let:

$w^+$ : be given the positive weight vector of the positive deviation variables,

$w^-$ : be given the negative weight vector of the negative deviation variables,

$d^+$ : is the positive deviation variable of the maximum goals,

$d^-$ : is the negative deviation variable of the minimum goals,

$x^o$ : be given feasible solution vector of LGP of multi-objective function,

$d^{+^o}, d^{-^o}$ : are given two vectors denoting the feasible positive deviation and the feasible negative deviation points of the max or min goals respectively.

The vectors feasible solution $x^o$ , $d^{+^o}$ , $d^{-^o}$ may or may not be vectors optimal solutions of LGP of multi-objective function with respect cost vectors $w^+, w^-$.

The inverse linear goal programming problem ILGP of a multi-objective function is as follows.

Consider the change of the cost vectors $w^+, w^-$ as less as possible such that the vectors feasible solution $x^o$ , $d^{+^o}$ , $d^{-^o}$ becomes an optimal solution of LGP of multi-objective function. Under the new cost vectors $\lambda^+, \lambda^-$ and $\| |\lambda^+ - w^+| + |\lambda^- - w^-| \|_p$ is minimal, where $\|.\|_p$ is some selected $L_p$-norm.

Inverse linear programming problem is a relatively new emerged one. In 1996, Zhang and Liu presented the earlier trial on this problem in the case of changing coefficient of the objective function [11, 12].

Ravindara K. Ahuja and James B. Orlin in 2001 described inverse optimization in the following manner [1, 2]:

Let $S$ denote the set of feasible solutions of an optimization problem $P$, let $c$ be a specified cost vector; and be a given feasible solution; that is, $P = \min\{cx : x \in S\}$. The inverse optimization problem is perturb the cost vector c to d so that $x^0$ is an optimal element of $P$ with respect to the cost vector $d$ , and $\|d - c\|_p$ is minimal, where $\|.\|_p$ denotes some selected $L_p$-norm .

Ahuja and Orlin considered inverse optimization problems under $L_1$-norm, where the objective is to minimize $\sum_{j\in J} w_j |d_j - c_j|$ and under the weighted $L_\infty$- norm, the objective is min max $\{w_j |d_j - c_j| : j \in J\}$. Here, $w_j$ are specified weights [2].

Ahuja and Orlin (1998a) also provided various references in the area of inverse optimization. Compile several applications, and developed algorithms to solve general inverse problem [2].

Abuja and Orlin (1998b) specialized their inverse optimization algorithms to inverse network flow problem [1].

Saif (2006) provided the inverse multi-objective linear programming problem within two cases [9]:

(i) Change in the sum of coefficient decision variables.
(ii) Change in at least one the right-hand side constraint.

Saif also presented a comparative study of the inverse optimization problems.
In this paper, we present the derivation of the inverse goal linear programming problem of multi-objective function, and its solution under the $L_1$-norm. Also, we present algorithms for solving the inverse versions to study assignment problem.

## 2. FORMULATING THE INVERSE LINEAR GOAL PROGRAMMING PROBLEM OF MULTI-OBJECTIVE FUNCTION

We consider the inverse version of the following linear goal programming problem of the multi-objective function.
**First:** given a multi-objective linear pro ramming problem as follow [3, 4]:
(MOLP)

$$\left.\begin{array}{l} \min z_1 = f_1(x_1, x_2, \dots, x_n), \\ \min z_2 = f_2(x_1, x_2, \dots, x_n), \\ \quad\vdots \\ \min z_k = f_k(x_1, x_2, \dots, x_n) \\ \qquad \text{such that} \\ \qquad g_j(x_1, x_2, \dots, x_n) \geq b_j \qquad (j = 1, 2, \dots, m), \\ \qquad x_1, x_2, \dots, x_n \geq 0, \end{array}\right\} \tag{1}$$

where:
$\underline{x} = (x_1, x_2, \dots, x_n)$ the n-vector of decision variables,
$Z_i \ (i = 1, 2, \dots, k)$ the vector of an objectives according to priority,
$f_i(\underline{x}) \ (i = 1, 2, \dots, k)$ the objective function associated with priority,
$g_j(\underline{x}) \ (j = 1, 2, \dots, m)$ the left hand side of the objective function,
$b_j \ (j = 1, 2, \dots, m)$ the right hand side of the constraint,
**Second:** the linear goal programming problem of the above problem is [5, 6]:
(LGMOP)

$$\left.\begin{aligned}
&\min \quad Z=\sum_{i=1}^{k}\left(w_i^{+}d_i^{+}+w_i^{-}d_i^{-}\right) && \text{(2a)}\\
&\qquad \text{such that}\\
&\sum_{l=1}^{n} c_{il}x_l + d_i^{+} - d_i^{-} = b_i \qquad (i=1,2,...,k), && \text{(2b)}\\
&\sum_{l=1}^{n} a_{jl}x_l \geq b_j \qquad (j=1,2,...,m), && \text{(2c)}\\
&x_l,\ \ d_i^{+},\ \ d_i^{-} \geq 0, && \text{(2d)}
\end{aligned}\right\} \quad (2)$$

where:

$f_i(\underline{x})=\sum_{l=1}^{n} c_{il}x_l \quad (i=1,2,...,k)$ the left-hand side of the linear objective function;

$g_j(\underline{x})=\sum_{l=1}^{n} a_{jl}x_l \quad (j=1,2,...,m)$ the left-hand side of the linear structural constraint;

$c_{il},\ a_{jl} \quad (i=1,2,...,k;\ j=1,2,...,m; l=1,2,...,n)$ constants,

$b_i = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_k \end{bmatrix}$ : the goals set by the decision maker for the objective $(i=1,2,...,k)$,

$b_j = \begin{bmatrix} b_1 \\ b \\ . \\ . \\ b_m \end{bmatrix}$ : the right-hand side bound on the structural constraint $(j=1,2,...,m)$,

$w_i^{+}, w_i^{-} \quad (i=1,2,...,k)$ given positive and negative weights vectors of the positive and negative deviation variables respectively,

$d_i^{+}, d_i^{-} \quad (i=1,2,...,k)$ tow vectors positive and negative deviation variables of the max or min goals, respectively.

**Third**: the dual of (LGMOP) is the following [4, 8]:

$$\left.\begin{array}{ll} max\ \ V=\sum_{i=1}^{k} b_i y_i + \sum_{j=k+1}^{k+m} b_j y_j & \\ \quad\quad such\ that & \\ \quad \sum_{i=1}^{k} c_{il} y_i + \sum_{j=k+1}^{k+m} a_{jl} y_j \le 0, & \\ \quad\quad y_i \le w_i^- & (i=1,2,...,k), \\ \quad\quad -y_i \le w_i^+ & (i=1,2,...,k), \\ \quad\quad y_i \quad : unrestricted\ in\ sign, & \end{array}\right\} \quad (3)$$

where:

$y_i\ ,y_j \quad (i=1,2,...,k\ \ ;\ \ j=k+1,k+2,...,k+m)$ are the dual variables with the constraint (2b), (2c) respectively.

**Fourth**: let $X^O$ given feasible solution vectors,

$d^{+o}\ ,\ d^{-o}$ given two vectors denote the feasible positive deviation and feasible negative deviation points of the max or min goals, respectively, of problem (2),

and let B denote the index set of binding constraints (2b), (2c) with respect to $X^O$ where:

$$B=\left\{ j\in J : \sum_{i=1}^{k} c_{il} x_l^o + d_l^{-o} - d_l^{+o} = b_i\ \ (i=1,2,...,k),\ \sum_{i=1}^{k} a_{jl} x_l^o = b_j \right\} \quad (4)$$

We only care about binding constraints and the dual decision variables y for binding constraints.

**Fifth**: using condition (4) in problem (3), we can formulate the inverse goal linear programming problem of multi-objective function under the $L_1$-norm according to the set of binding constraint B as follows.

(I LGMOP)

$$\left.\begin{array}{lll} min\ \ Z_{inv}=\sum_{i=1}^{k}\left[\left|w_i^+ +\lambda_i^+\right| + \left|w_i^- +\lambda_i^-\right|\right] = \sum_{l=1}^{n}\left(\theta_l^+ + \theta_l^-\right) & & (5a) \\ \quad\quad such\ that & & \\ \quad \sum_{l\in B} c_{il} y_i + \sum_{j=k+1\in B} a_{jl} y_j - \left(\theta_l^+ + \theta_l^-\right) = 0, & & (5b) \\ \quad y_i - \theta_l^- = w_i^- & (i\in B), & (5c) \\ \quad -y_i - \theta_l^+ = w_i^+ & (i\in B), & (5d) \\ \quad y_i\ \ :\ unrestricted\ in\ sign & (i\in B), & (5e) \\ \quad y_j \ge 0\ \ ,\forall\ j\in B\ \ ;\ \ \theta_l^+,\theta_l^- \ge 0 & (i\in B), & (5f) \end{array}\right\} \quad (5)$$

## 3. THE SOLUTION OF INVERSE LINEAR GOAL PROGRAMMING PROBLEM OF MULTI-OBJECTIVE FUNCTION UNDER $L_1$ -NORM

Let us solve the inverse linear goal programming problem of multi-objective in problem (5) by following steps:

**Step (1):** we use the equation (4) to find the equivalent formulation for the problem (5):

$$\left.\begin{aligned}
&min \quad Z'_{inv} = \sum_{i=1}^{k}\left( w_i^{\,+} d_i^{+'} + w_i^{\,-} d_i^{-'} \right) \\
&\qquad\qquad such\ that \\
&\sum_{i\in B} c_{il}\, y_i^{'} + d_i^{-'} - d_i^{+'} = b_{i\in B}\,, \\
&\sum_{i\in B} a_{jl}\, y_j^{'} \ \geq\ b_{j\in B}\,, \\
&d_i^{-'}, d_i^{+'}, y_i^{'}, y_j^{'},\ w_i^{+}, w_i^{-} \ \geq 0,
\end{aligned}\right\} \tag{6}$$

where:

$b_{i\in B}$ the value $b_i$ according to B,

$b_{j\in B}$ the value $b_j$ according to B,

We note that the inactive constraints are cancelled.

**Step (2):** we find the dual problem of the problem in equation (6):

$$\left.\begin{aligned}
&min \quad V'_{inv} = \sum_{i=1}^{k} b_{i\in B}\, x'_i + \sum_{j=k+1}^{k+m} b_{j\in B}\, x'_j \\
&\qquad\qquad such\ that \\
&\sum_{l\in B} c_{li}\, x'_i + \sum_{j=k+1\in B} a_{lj}\, x'_j \ \leq\ 0, \\
&x'_i \leq w_i^{-} \qquad\qquad ( i = 1,2,\dots, k ), \\
&- x'_i \leq w_i^{+} \qquad\qquad ( i = 1,2,\dots, k ), \\
&w_i^{+}, w_i^{-} \ \geq 0\ ,\ x'_j \geq 0\ , \quad x'_i\ :\ unrestricted\ in\ sign
\end{aligned}\right\} \tag{7}$$

**Step (3):** we find $\left( w_i^{+} \right)^{y}$ , $\left( w_i^{-} \right)^{y}$

$$\left( w_i^{+} \right)^{y} = w_i^{+} - \left[ \sum_{l\in b} c_{li}\, x'_l + \sum_{j\in b} a_{ji}\, x'_j - x'_i \right], \tag{8}$$

$$\left( w_i^{-} \right)^{y} = w_i^{-} - \left[ \sum_{l\in b} c_{li}\, x'_l + \sum_{j\in b} a_{ji}\, x'_j + x'_i \right], \tag{9}$$

**Step (4):** find the solution of the inverse goal linear programming of multi-objective (optimal value for weight $\lambda^{+}$ , $\lambda^{-}$ ) as follow:

$$\lambda_i^{+} = \begin{cases} w_i^{+} - \left| \left( w_i^{+} \right)^{y} \right| & , \quad \left( w_i^{+} \right)^{y} > 0 \\ w_i^{+} + \left| \left( w_i^{+} \right)^{y} \right| & , \quad \left( w_i^{+} \right)^{y} < 0 \\ w_i^{+} & , \quad \left( w_i^{+} \right)^{y} = 0 \end{cases} \tag{10}$$

$$\lambda_i^- = \begin{cases} w_i^- - \left|(w_i^-)^y\right| & , \quad (w_i^-)^y > 0 \\ w_i^- + \left|(w_i^-)^y\right| & , \quad (w_i^-)^y < 0 \\ w_i^- & , \quad (w_i^-)^y = 0 \end{cases} \tag{11}$$

# 4. APPLICATION OF INVERSE LINEAR GOAL PROGRAMMING PROBLEM OF A MULTI-OBJECTIVE FUNCTION IN AN ASSIGNMENT PROBLEM

The previous results for I.L.G.P of multi-objective can be used to study the inverse version of the assignment problem.

The assignment problem is the following multi-objective linear programming Problem [7, 8]:

$$\left.\begin{aligned} & \min Z_L = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^l x_{ij} && (l = 1,2,\dots,k) \\ & \quad s.t. \\ & \sum_{j=1}^{n} x_{ij} = 1 && (i = 1,2,\dots,n), \\ & \sum_{i=1}^{n} x_{ij} = 1 && (j = 1,2,\dots,n), \\ & c_{ij}^l \geq 0 && (l = 1,2,\dots,n \quad ; \quad i = 1,2,\dots,n), \\ & x_{ij} \geq 0 && (l = 1,2,\dots,n \quad ; \quad i = 1,2,\dots,n), \end{aligned}\right\} \tag{12}$$

Where:

$$x_{ij} = \begin{cases} 1 & \text{if employee } i \text{ is assigned to task } j, \\ 0 & \text{otherwise} \end{cases}$$

$c_{ij}^l$ : is denoting to coefficient decision matrix form rank j

The goal programming of last assignment problem in problem (12) is shown as follows [4]:

$$\left.\begin{aligned} & \min\ Z = \sum_{i=1}^{n}\left(w_{ij}^{+} d_{ij}^{+} + w_{ij}^{-} d_{ij}^{-}\right) && (j = 1,2,\dots,n) \\ & \quad s.t \\ & \sum_{j=1}^{n} c_{ij}^l x_{ij} + d_{ij}^{+} - d_{ij}^{-} = b_{ii} && (i = 1,2,\dots,n \quad ; l = 1,2,\dots,k), \\ & \sum_{i=1}^{n} x_{ij} = 1 && (j = 1,2,\dots,n), \\ & \sum_{j=1}^{n} x_{ij} = 1 && (i = 1,2,\dots,n), \\ & x_{ij}, \quad d_{ij}^{+}, \quad d_{ij}^{-} \geq 0, \end{aligned}\right\} \tag{13}$$

The dual problem of problem in (13) as follow [7, 8]:

$$\left.\begin{aligned}
&max\ \ V=\sum_{i=1}^{n}b_{ii}y_{ii}+\sum_{j=1}^{n}y_{jj}+\sum_{i=1}^{n}y_{ii}\\
&\qquad s.t\\
&\sum_{i=1}^{n}c_{ij}^{l}y_{ii}+\sum_{j=1}^{n}y_{jj}+\sum_{i=1}^{n}y_{ii}\le 0 \qquad (l=1,2,\dots,k).\\
&\qquad y_{ii}\le w_{ii}^{-} \qquad\qquad (i=1,2,\dots,k).\\
&\quad -y_{ii}\le w_{ii}^{+} \qquad\qquad (i=1,2,\dots,k),\\
&\qquad y_{ii}\quad : unrestricted\ in\ sign
\end{aligned}\right\} \tag{14}$$

The inverse assignment problem it means change in $\left(w_{ij}^{+},w_{ij}^{-}\right)$ such that the feasible assignment becomes an optimal assignment.

Assume that the feasible solution $X^{0}$ either 1 or 0, by use problem (5), with respect to $X^{O}$ then the inverse assignment problem as linear goal multi-objective programming problem under $L_1$-Norm as follow:

$$\left.\begin{aligned}
&min\ \ U_{inv}=\sum_{j=1}^{n}\left[\left|w_{ij}^{+}-\lambda_{ij}^{+}\right|+\left|w_{ij}^{-}-\lambda_{ij}^{-}\right|\right]=\sum_{i=1}^{n}\left|\theta_{ij}^{+}+\theta_{ij}^{-}\right| \qquad (j=1,2,\dots,n),\\
&\qquad s.t\\
&\sum_{i=1}^{n}c_{ij}^{l}y_{ii}+\sum_{j=1}^{n}y_{jj}+\sum_{i=1}^{n}y_{ii}-\left(\theta_{ij}^{+}+\theta_{ij}^{-}\right)=0 \qquad (j=1,2,\dots,n),\\
&\qquad y_{ii}-\theta_{ij}^{-}=w_{ii}^{-} \qquad\qquad ,\forall\ i\in J^{-}.\\
&\quad -y_{ii}-\theta_{ij}^{+}=w_{ii}^{+} \qquad\qquad ,\forall\ i\in J^{+}.\\
&\qquad y_{ii}\quad : unrestricted\ in\ sign \quad (i=1,2,\dots,n),\\
&\qquad y_{jj}\ge 0\ \ ,\ \ \theta_{ij}^{+}\ \ ;\ \ \theta_{ij}^{-}\ge 0 \qquad (j=1,2,\dots,n),
\end{aligned}\right\} \tag{15}$$

Where:

$$J^{-}=\{(i,\ \ j):X_{ij}^{0}=1\}.$$

$$J^{+}=\{(i,\ \ j):X_{ij}^{0}=0\}$$

Using the result in equations (10), (11) gives us the following optimal cost vectors $\lambda_{ij}^{+*},\lambda_{ij}^{-*}$ for the inverse assignment problem:

$$\lambda_{ij}^{+*}=\left\{\begin{array}{ll} w_{ij}^{+}-\left|w_{ij}^{+}-\left[\sum_{(i,\ j)\in J^{-}}c_{ij}y_{ij}+\sum_{(i,\ j)\in J^{-}}a_{ji}y_{ij}-y_{ij}\right]\right| & ,\ \ i=j\\ 0 & ,\ \ otherwise\end{array}\right\}$$

$$\lambda_{ij}^{-*}=\left\{\begin{array}{ll} w_{ij}^{-}-\left|w_{ij}^{-}-\left[\sum_{(i,\ j)\in J^{-}}c_{ij}y_{ij}+\sum_{i,j)\in J^{-}}a_{ji}y_{ij}+y_{ij}\right]\right| & ,\ \ i=j\\ 0 & ,\ \ otherwise\end{array}\right\}$$

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# المشكلة العكسية لمشكلة البرمجة الخطية الصدفية وحيدة الهدف ومتعددة الأهداف ذات الأوزان

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**ملخص**

يتناول البحث إيجاد المشكلة العكسية لمشكلة البرمجة الخطية الهدفية الوحيدة والمتعددة الأهداف ذات الأوزان وذلك بالتغيير في معاملات دالة الهدف أو في الأوزان الذي يجعل من أي حل ممكن حلاً أمثلاً ومن ثم حل المشكلة العكسية باستخدام طرق حل مشاكل البرمجة الخطية وتطبيقاتها في الشبكات