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Zeta Function for Commuting Matrix Shift of Finite Type

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ABSTRACT

In this paper we show two-dimensional shifts of finite type whose transition matrices commutes and such that the entries of their product also consist of 0's and 1's. In particular, the form of associated zeta function of the two-dimensional shift was obtained. Unlike in the one-dimensional shifts of finite type; we gather that zeta functions for them have not closed form. This result derived by obtaining a formula for the number of periodic points of the subshift of finite type. This formula is then incorporated in the corresponding zeta function.

1. INTRODUCTION

There has been great deal of interest has been concerning theoretical issues on a class of dynamical systems known as the higher-dimensional subshifts of finite type. In part to in the one-dimensional analogue of the aforementioned systems example(Johnson et al.,19998) and (schmidt,2000). It is well-known that each subshift of finite type can be defined via their corresponding transition matrices. And probably one of the first papers to exploit this situation, as far as higher-dimensional subshift of finite type is concerned,(markley et al.,1981)and Al-Refaei.2011). In this paper, we shall also assume that the associated transition matrices commutes but with the extra assumption that the product of these matrices also have 0-1 entries, (Noorani,M.1996)and Al-Refaei,A.2011). Our main goal is to study the form of an associated zeta function. In this paper, it shown that, unlike in the one-dimensional case, the corresponding function for the higher-dimensional analogue of this dynamical system may not have a closed form. To arrive at this conclusion we shall present a formula for the number of the periodic points of these subshifts, (Noorani,M.2002). We shall then invoke the zeta function introduced by (Lind,D.1995).where the aforementioned formula is then incorporated. Along the way we also recover some of the results proposed in (Lind,D.1995). To fix ideas, we shall concentrate entirely in the two-dimensional case.

2. PERIODIC POINTS

We are interested in counting the number of beriodic points for a given matrix subshifts. It is convenient to introduce the following notation which is due to (Lind,D. 1995). Let $S=\{0,1,.....,k-1\}$ for some fix positive integer *k* and *H, V* is two $k\times k$ 0-1 matrices, Also let *X* be the set

$$X=\left\{x=(x_n)\in S^{\mathbb{Z}^2}:H\left(x_n,x_{n+e_1}\right)=V\left(x_n,x_{n+e_2}\right)=1,\forall n\in\mathbb{Z}^2\right\}$$

Where e_i's $i=1,2$ are the standard basis vectors for $\mathbb{Z}^2$.

The set *X* together with the shift map $\sigma_n:X\to X$ defined by $\left(\sigma_n(x)\right)_a=x_{n+a}\ a,n\in\mathbb{Z}^2$ is called the subshift of finite type with respect to the transition matrices *H* and *V*.

We are interested in counting the number of periodic points for a given matrix subshifts.

It is convenient to introduce the following notions which are due to Lind .Let the set of all finite-index subgroups of $\mathbb{Z}^2$ by $\mathfrak{I}_2$ .For $L\in\mathfrak{I}_2$, let $Fix_L=card\left\{x\in X:\sigma_n(x)=x\forall n\in L\right\}$, denote the number of points in *X* which are fixed by elements of *L*. As is in (Lind,D.1995). We are need the following classical result from the theory of integer matrices,(Manning,1971).

Proposition 1

Every finite-index subgroup of $\mathbb{Z}^d$ can be uniquely represented as the image of $\mathbb{Z}^d$ under a matrix having the form

$$\begin{pmatrix} r_1 & q_{12} & q_{13} & \cdots & q_{1d} \\ 0 & r_2 & q_{23} & \cdots & q_{2d} \\ 0 & 0 & r_3 & \cdots & q_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & r_d \end{pmatrix}$$

Where $r_i\geq 1\forall i=1,2,...,d\ \&\ o\leq q_{ij}\leq r_i-1,\forall i+1\leq j\leq d.$

From the above general result we have the following parameterization $\mathfrak{I}_2^{\ *}$ of $\mathfrak{I}_2$,

$$\mathfrak{I}_2^{\ *}=\left\{\begin{pmatrix} r_1 & q \\ 0 & r_2 \end{pmatrix}\mathbb{Z}^2:r_1,r_2\geq 1,0\leq q\leq r_1-1\right\}.$$

From this parameterization, we have the following proposition.

Proposition 2

Let *X* be a 2-dimentional matrix subshift with transitions matrices *H,V* such that $HV=VH$ and *HV* is a *0-1matrix*. Then for each $L\in\mathfrak{I}^*{}_2$ where

$L = \begin{pmatrix} r_1 & q \\ 0 & r_2 \end{pmatrix}\mathbb{Z}^2$, we have

$$Fix_L = Trace\left(H^q V^{r_2}\right).$$

3 .ZETA FUNCTIONS

The following was introduced by (Lind,D. 1995). The zeta function associated with a 2-dimensionsl subshift of finite type is given as

$$\zeta(z) = \exp\left(\sum_L \frac{Fix_L}{[L]} z^{[L]}\right)$$

Where the sum is taken over all subgroups *L* of z^2 with finite index and *[L]* denotes the index of *L*. One recovers the *1*-dimensional zeta function of (Artin,M.1965) when one considers finite-index subgroups of z^2 which are given by the cosets $\{nz : n \geq 1\}$ Of course the above definition also holds for arbitrary *d-dimensional subshift of finite type*. The zeta function is clearly conjugacy invariant because the number of point of each period is preserved. For $A \in GL(d, Z)$, then z^2*-shift of finite type* σ^A can be defined as $\left(\sigma^A\right)^n = \sigma^{An}$.

Proposition 3

Let σ be a z^2*-shift of finite type* and $A \in GL(d, \mathbb{Z})$. Then $\zeta_\sigma(z) = \zeta_{\sigma^A}(z)$.

Proof:

For $L \in \Im_d$ let $AL = \{A_n : n \in L \}$

The $L \leftrightarrow AL$ is a bijection by $g : L \leftrightarrow AL$ & g(n)=An,

$\forall n \in L, A \in GL(d, \mathbb{Z}) \Rightarrow [L] = \left|\mathbb{Z}^2 / L\right| = \left|\mathbb{Z}^2 / AL\right| = [AL]$

Also $x \in \{x \in X : \sigma_n(x) = x \forall n \in L \}$

Iff $\sigma_n(x) = x \forall n \in L$,and *x* is fixed point of σ_n in *X* and

$x \in \{x \in X : \sigma^{An}(x) = x \forall An \in AL \}$

$\Rightarrow \left|\{x \in X : \sigma^n(x) = x \forall n \in L \}\right| = \left|\{x \in X : (\sigma^A)^n(x) = x \forall An \in AL \}\right|$

$\Rightarrow P_{AL}(\sigma) = P_L(\sigma^A) = P_L(\sigma)$. Then

$$\zeta_{\sigma^A}(z) = \exp\left(\sum_{L \in \Im_d} \frac{P_L(\sigma^A)}{[AL]} z^{[AL]}\right) = \exp\left(\sum_{L \in \Im_d} \frac{P_L(\sigma)}{[L]} z^{[L]}\right) = \zeta_\sigma(z) \cdot$$

The following examples will illustrate how to calculate the zeta functions when we get the periodic point.

Example1

Let σ be σ *-invariant* of finite type on single point *X ={x}*

Then *X* is compact space, and $\sigma_n(x) = x \,\forall n \in L \,\&\, P_L \quad (\sigma) = 1 \forall L \in \Im_2$

In $\mathbb{Z}^2$ let $\Im_2^* = \left\{ \begin{pmatrix} r_1 & q \\ 0 & r_2 \end{pmatrix} \mathbb{Z}^2 : r_1, r_2 \geq 1, 0 \leq q \leq r_1 - 1 \right\}$. Then the zeta function of σ is

$$\zeta_\sigma(z) = \exp\left(\sum_{L \in \Im_2^\bullet} \frac{P_L(\sigma)}{[L]} z^{[L]} \right) = \exp\left(\sum_{L \in \Im^\bullet_2} \frac{1}{r_1 r_2} z^{r_1 r_2} \right)$$

$$= \exp\left(\sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \sum_{q=0}^{\infty} \frac{1}{r_1 r_2} z^{r_1 r_2} \right) = \exp\left(\sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \frac{1}{r_1 r_2} z^{r_1 r_2} \left((r_1 - 1) - 0 + 1 \right) \right)$$

$$= \quad \exp\left(\sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \frac{1}{r_1 r_1} z^{r_1 r} r_1 \right) = \exp\left(\sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \frac{\left(z^{r_1} \right)^{r_2}}{r_2} \right)$$

$$= \exp\left(\sum_{r_1=1}^{\infty} -\log\left(1 - z^{r_1} \right) \right) = \exp\left(\sum_{r_1=1}^{\infty} \log\left(\frac{1}{1 - z^{r_1}} \right) \right)$$

$$= \prod_{r_1=1}^{\infty} \left(\exp\left(\log\left(\frac{1}{1 - z^{r_1}} \right) \right) \right) = \prod_{r_1=1}^{\infty} \frac{1}{1 - z^{r_1}} \cdot$$

Example 2

Let σ be the *full* $\mathbb{Z}^2$ *k-shift*, then

$$\left(X = \{0, 1, ..., k-1\}^{\mathbb{Z}^2}, \left(\sigma^n(x) \right)_a = x_{a+n} \,\forall x \in X \,\forall n, a \in L \right)$$

Then $\forall L \in \Im_2 \,\&\, P_L(\sigma) = K^{[L]} = K^{r_1 r_2}$

So the zeta function is $\exp\left(\sum_{L \in \Im_2} \frac{K^{[L]}}{[L]} z^{[L]} \right)$

$$= \exp\left(\sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \sum_{q=0}^{\infty} \frac{K^{r_1 r_2}}{r_1 r_2} z^{r_1 r_2} \right) = \exp\left(\sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \frac{\left((Kz)^{r_1} \right)^{r_2}}{r_2} \right)$$

$$= exp\left(\sum_{r_1=1}^{\infty} -\log\left(1 - (Kz)^{r_1} \right) \right) = \prod_{r_1=1}^{\infty} \frac{1}{1 - (Kz)^{r_1}}.$$

4. RESULTS

From the above propositions, examples and by (Al-Refaei,A.2011) we get this Theorem which will be help us to get the zeta function for any periodic point that we jet it by transition matrix.

Theorem

Let *X=X(H,V)* be *a 2-dimensional matrix subshift* such that *HV=VH* and *HV* is a *0-1 matrix*. Then the zeta function of *X* is formally given by

$\zeta(z)=\prod_{i=1}^{2}\prod_{j}^{\infty}\left(1-\mu_i z^j\right)^{-\lambda_i^j}$ where $\lambda_i^j=\left(\lambda_i+\lambda_i^2+...+\lambda_i^j\right)\Big/ j$ $and\,\lambda_i,\mu_{i,,i=12}$ are the eigenvalue of *H* and *V* respectively .

Proof :

Since *H* and *V* commutes, one can simultaneously 'diagonals' them via a n*on-singular matrix R* (see, for e.g, (Jacobson 1953)) so that

$$Fix_L=Trace\left(H^qV^{r_2}\right)=Trace\left(RH^qR^{-1}RV^{r_2}R^{-1}\right)$$

$$=Trace\left(diag\left(\lambda_1^q,\lambda_2^q\right)\right)\left(diag\left(\mu_1^{r_2},\mu_2^{r_2}\right)\right)$$

$$=Trace\left(diag\left(\lambda_1^q\mu_1^{r_2},\lambda_2^q\mu_2^{r_2}\right)\right)=\lambda_1^q\mu_1^{r_2}+\lambda_2^q\mu_2^{r_2}$$

Hence

$$\left(\exp\sum_L\frac{Fix_L}{[L]}\right)=\exp(\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{q=1}^{r-1_1}\frac{Trace\left(H^qV^{r_2}\right)}{r_1r_2}z^{r_1r_2})$$

$$=\exp\left(\sum_{i=1}^{2}\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{q=1}^{r_{1-1}}\frac{\lambda_i^q\mu_i^{r_2}}{r_1r_2}z^{r_1r_2}\right)$$

we see that

$$\sum_{r_2=1}^{\infty}\sum_{q=1}^{r_1-1}\frac{\lambda_i^q\mu_i^{r_2}}{r_1r_2}z^{r_1r_2}=-\lambda_i^{r_1}\log\left(1-\mu_i z^{r_1}\right)$$

Therefore,

$$\left(\sum_L\frac{Fix_L}{[L]}\right)=\exp\left(\sum_{i=1}^{2}\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{q=1}^{r_{1-1}}\frac{\lambda_i^q\mu_i^{r_2}}{r_1r_2}z^{r_1r_2}\right)=\exp\sum_{i=1}^{2}\sum_{r_1=1}^{\infty}\log\left(1-\mu_i z^{r_1}\right)^{-\lambda_i^{r_1}}$$

$$=\prod_{i=1}^{2}\prod_{r_1=1}^{\infty}\left(1-\mu_i z^{r_1}\right)^{-\lambda_i^{r_1}}$$

In particular, we have

$$\zeta(z)=\exp\left(\sum_L\frac{Fix_L}{L}z^{[L]}\right)=\exp\left(\log\prod_{i=1}^{2}\prod_{r_1=1}^{\infty}\left(1-\mu_i z^{r_1}\right)^{-\lambda_i^{r_1}}\right)=\prod_{i=1}^{2}\prod_{r_1=1}^{\infty}\left(1-\mu_i z^{r_1}\right)^{-\lambda_i^{r_1}}.$$

5. CONCLUSIONS

This papers focused on properties for shift of finite type exactly zeta function for subshift in dimension two, to present various method to calculate periodic point and zeta function for two-shift of finite type using subgroup and transition matrix .

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إستخدام دالة الزيتا
لحساب عمليات نقل المصفوفة من النوع المحدود

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ملخص

في هذه الورقة نعرض ثنائية الأبعاد تحولات من نوع محدود والتي تمر بمرحلة انتقالية للمصفوفات المحولة ومثل هذه الإدخالات من ناتجها تتألف من الأرقام 0 و 1. على وجه الخصوص، تم الحصول عليها على شكل دالة زيتا يرتبط بها من التحول ثنائية الأبعاد. وخلافا للتحولات ذات بعد واحد من نوع محدد، ان تجتمع دالة زيتا بالنسبة لها ليس لها شكل مغلق. هذه النتيجة التي نحصل عليها للحصول على صيغة لعدد من النقاط الدورية للتحول من نوع محدد. ثم أدرج هذه الصيغة في دالة زيتا المقابلة.