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On A Class of Locally Convex Involution Algebras

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ABSTRACT

In this paper we introduced a class of locally convex involution algebras called MBG*-algebras as generalized of GB*-algebras was introduced by Allan. We obtain some results on this class and established a necessary and sufficient conditions for a commutative MGB*-algebra to be symmetric.
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1. INTRODUCTION

Since the theory of Banach algebras has undergone considerable development, the concepts of locally convex *-algebras and pseudo-complete locally convex *-algebras (PL*-algebras) was introduced by Allan [1] and as generalizations of normed *-algebras and in particular Banach *-algebras. In [2], Allan defined a class of locally convex *-algebras called GB*-algebras as a generalization of B*-algebras. In this paper we introduced a new class of locally convex *-algebras called MGB*-algebras (definition (3.3)) as a modification of GB*-algebras. This class is more general of the definition of GB*-algebras introduced by Allan [2]. The required definitions and results from books and research papers (cited in the bibliography) are listed in §2.
In §3 we obtain some results on locally convex MBG*-algebras and establish a necessary and sufficient conditions for a commutative MGB*-algebra to be symmetric, Theorem (3.11).

2. LOCALLY CONVEX*-ALGEBRAS

We shall use the basic terminologies of definitions and results of this section from books and research papers cited in the bibliography.
DEFINITION 2.1 [7] A **-algebra* is an associative linear algebra A over the complex filed $\mathbb{C}$ with an involution * satisfying the usual axioms

(i) $(\lambda x+\mu y)^* = \bar{\lambda}x^*+\bar{\mu}y^* \quad (x,y\in A;\lambda,\mu\in C)$;
(ii) $(xy)^* = y^*x^* \quad (x,y\in A)$;

$(iii)\ x^{**} = x \quad (x \in A)$.

An element $x \in A$ will be called hermitian if $x^* = x$ and normal if $x^*x = xx^*$. The set of all hermitian elements of *A* will be denoted by A_h.

DEFINITION 2.2 [1] A *locally convex algebra* is an associative linear algebra *A* over the complex filed $\mathbb{C}$ equipped with a topology τ such that

(i) (A, τ) is a Hausdorff locally convex topological vector space ;

(ii) for any element $x_0 \in A$ the maps $x \to x_0 x$ and $x \to x x_0$ of *A* into itself are continuous.

DEFINITION 2.3 [1] Let *E* be a locally convex space. A subset *B* of *E* is said to be *bounded* if for each neighborhood *V* of the origin, there is $\lambda > 0$ such that $B \subseteq \lambda V$.

DEFINITION 2.4[1] Let *A* be a locally convex algebra. An element $x \in A$ is *bounded* if and only if for some non-zero $\lambda \in \mathbb{C}$ the set $\{(\lambda x)^n : n = 1,2, \dots\}$ is a bounded subset of *A*. The set of all bounded elements of *A* will be denoted by A_0 .

REMARK 2.5 Every element of a normed algebra is bounded so for each normed algebra *A*, $A = A_0$.

NOTATION 2.6 If *A* is a locally convex algebra, then $\mathcal{B}_1$ will denote the collection of all subsets *B* of *A* such that

(i) *B* is absolutely convex and $B^2 \subseteq B$,

(ii) *B* is bounded and closed.

For each $B \in \mathcal{B}_1$, let *A*(*B*) denote the sub-algebra of *A* generated by *B*. Then from (i) and (ii)

$$A(B) = \{\lambda x : \lambda \in \mathbb{C}, x \in B\}.$$

And the Minkowski functional

$$\| x \|_B = \inf\{\lambda > 0 : x \in \lambda B\} \quad (x \in A(B)),$$

defines a norm on $A(B)$, which make $A(B)$ a normed algebra. Unless mention is made to the contrary, it will always be assumed that $A(B)$ carries the topology induced by this norm.

DEFINITION 2.7[1] The locally convex algebra *A* is called *pseudo-complete algebra* if and only if each normed algebra $A(B)$ $(B \in \mathcal{B}_1)$ is complete (Banach algebra) and denoted by PL-algebra.

DEFINITION 2.8 A subcollection $\mathcal{B}_2$ of $\mathcal{B}_1$ is said to be *basic* in $\mathcal{B}_1$ if for every B_1 in $\mathcal{B}_1$ there is some B_2 in $\mathcal{B}_2$ such that $B_1 \subseteq B_2$.

PROPOSITION 2.9 Let *A* be a locally convex algebra and let $\mathcal{B}_2$ be any basic subcollection in $\mathcal{B}_1$. Then

$$A_0 = \cup \{A(B) : B \in \mathcal{B}_2\}.$$

***Proof*.** See [1] Proposition(2.4).

PROPOSITION 2.10 Let *A* be a locally convex algebra. If *A* is sequentially complete then *A* is pseudo-complete but not conversely.

***Proof*.** See [1] Proposition(2.6) and example1.

DEFINITION 2.11 If *A* is a locally convex algebra, then the *radius of boundedness*, $\beta(x)$, of *x* is defined by

$$\beta(x) = \inf\{\lambda > 0: \{(\lambda^{-1}x)^n : n = 1,2,\dots\}\ is\ bounded\}$$

with the convention $inf\,\phi = +\infty$.

DEFINITION 2.12 (i) Let *A* be a locally convex algebra with identity *e* and let $x \in A$. The *spectrum* of *x* in *A* denoted by $\sigma_A(x)\ (or\ just\ \sigma(x))$, is that subset of $\mathbb{C}^*$ (The extended of $\mathbb{C}$) defined as follows

(a) for $\lambda \neq \infty, \lambda \in \sigma(x)$ if and only if $\lambda e - x$ has no inverse belonging to A_O (i.e. has no bounded inverse),

(b) $\infty \in \sigma(x)$ if and only if $x \notin A_O$.

(ii) If *A* has no identity, we define $\sigma_A(x) = \sigma_{A_e}(x)$,

where A_e is the unit adjoint . By [1] *A* is PL-algebra if and only if A_e is PL-algebra.

DEFINITION 2.13 Let *A* be a locally convex algebra with identity *e* and let $x \in A$. Then the *spectral radius* of *x* in *A* denoted by $r_A(x)(or\ just\ r(x))$ will be defined by

$$r_A(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

THEOREM 2.14 Let *A* be a locally convex algebra and let $x \in A$. Then $\beta(x) \leq r(x)$ and if *A is* PL-algebra then $\beta(x) = r(x)$.

Proof. See [1] Theorem(3.12) .

DEFINITION 2.15 A locally convex *-algebra is a locally convex algebra with a continuous involution. And a pseudo-complete locally convex *-algebra (PL*-algebra) is a PL-algebra with a continuous involution.

PROPOSITION 2.16 A closed subalgebra of pseudo-complete algebra (resp. *-algebra) is itself pseudo-complete algebra (resp. *-algebra).

Proof. See [1] Proposition(2.8).

PROPOSITION 2.17 If *A* is a commutative pseudo-complete algebra (resp. *-algebra) then A_O is a subalgebra (resp. *-algebra) of *A*.

Proof. See Corollary(2.11) in [1].

DEFINITION 2.18 Let *A* be a locally convex*-algebra. The involution in *A* is said to be hermitian if for each $h \in A_h$, $\sigma_A(h)$ is real.

PROPOSITION 2.19 Let *A* be a PL*-algebra with identity *e* . If $x \in A$ such that $\beta(x) < 1$ then there exists $y \in A$ such that $2y - y^2 = x$, furthermore if $x \in A_h$ then also $y \in A_h$.

Proof. See [6] Lemma1.

REMARK 2.20 If *A* is a complex *-algebra then every element has a unique representation in the form *x*=*h*+*ik* where *h* and *k* are hermitian. In fact $h = \frac{x+x^*}{2}$ and $k = \frac{x-x^*}{2i}$.

THEOREM 2.21 Let *A* be a real or complex algebra and let P(*x*) be any polynomial with coefficients in the field of the scalars of *A* and with constant term zero in case *A* does not have an identity. Then, for every $x \in A, P(x) \in A$ and $P(\sigma_A(x)) = \sigma_A(P(x))$.

Proof. See [7] Theorem(1.6.10).

PROPOSITION 2.22 Let *A* be a locally convex algebra. Then

(i)Any subset or any scalar multiple of a bounded set is bounded,

(ii)Any finite union or sum of bounded sets is bounded,

(iii)The image by a continues linear mapping of a bounded set is bounded,
(iv)The closure, convex envelope and absolutely convex envelope of a bounded set are bounded.

Note: The following propositions (Proposition 2.23 and Proposition 2.24) was we obtained in another work (as a paper) and we need them in this paper, so we also give him prove here.

PROPOSITION 2.23 If *A* is a locally convex*-algebra with identity *e* .Then

(i) If $x \in A_O$,then x^2 and $x - x^2 \in A_O$,

(ii) If $x^2 \in A_O$, then $x \in A_O$.

Proof. (i) let $x \in A_O$, then there exists $\lambda \neq 0$ such that

$$S_1 = \{(\lambda x)^n : n = 1,2,\dots\}$$

is a bounded set in *A* and $S_1 \subseteq A_O$. Let

$$S_2 = \{(\lambda^2 x^2)^n : n = 1,2,\dots\}$$

It is clear that $S_2 \subseteq S_1$. Since every subset of a bounded set is bounded, then S_2 is a bounded set and thus $x^2 \in A_O$. Also $\lambda^{-1} S_1$ and $\lambda^{-2} S_2$ are bounded sets, then by Proposition(2.22) $\lambda^{-1} S_1 - \lambda^{-2} S_2$ is a bounded set. Since $x - x^2 \in \lambda^{-1} S_1 - \lambda^{-2} S_2$, then $x - x^2 \in A_0$.

(ii)Suppose $x^2 \in A_O$. Then there exists $\lambda \neq 0$ such that the set

$$S_1 = \{(\lambda^2 x^2)^n : n = 1,2,\dots\}$$

is bounded. Let $S_2 = S_1 \cup \lambda x S_1 \cup \{\lambda x\}$. Since *A* is a locally convex*-algebra then for each $y \in A$ the function

$$f_y : A \to A, \qquad f_y(x) = yx$$

is a continuous function . Since S_1 and $\{e\}$ are bounded sets in *A* and by Proposition(2.22) the image by a continuous function of a bounded set is bounded, and since

$$f_e(S_1) = S_1,\ f_{\lambda x}(S_1) = \lambda x S_1, and\ \ f_{\lambda x}(\{e\}) = \{\lambda x\}.$$

then S_1, $\lambda x S_1$ and $\{\lambda x\}$ are bounded sets in *A*. Since any finite union of bounded sets is a bounded set, therefore $S_2 = S_1 \cup \lambda x S_1 \cup \{\lambda x\}$ is a bounded set in *A.* since

$$\{(\lambda x)^n : n = 1,2,\dots\} \subseteq S_2.$$

Then the set
$\{(\lambda x)^n : n = 1,2,\dots\}$ is a bounded set . Therefore $x \in A_O$. □

PROPOSITION 2.24 Let *A* be a locally convex*-algebra with identity *e*. If the involution in *A* is hermitian then for every $h \in A_h$, $e + h^2$ has a bounded inverse. Further, the element $h(e + h^2)^{-1}$ is also bounded.

Proof. Assume that the involution in *A* is hermitian and let $h \in A_h$, then $\sigma_A(h)$ is real and by Theorem(2.21), $\sigma_A(h^2) = (\sigma_A(h))^2$. Then $\sigma_A(h^2)$ is non-negative real and hence $-1 \notin \sigma_A(h^2)$. Thus $e + h^2$ has a bounded inverse. Put $u = h(e + h^2)^{-1}$. Then $u^2 = h^2(e + h^2)^{-2} = (e + h^2)^{-1} - (e + h^2)^{-2}$. Since $(e + h^2)^{-1} \in A_0$ then by Proposition(2.23) u^2 and then $u \in A_0$.

3. MGB *-ALGEBRAS

In this section we introduced a new class of locally convex algebras with an envolution so-called MGB*-algebras, as a modification of GB*-algebras was introduced in [2] by Allan, and show that each commutative MGB*-algebra is symmetric if and only if the involution is hermitian.

Notation 3.1 (a) If *A* is a locally convex *-algebra with identity *e*, as [2] then $\mathcal{B}'$ will denote the collection of subsets *B* of *A* satisfying:

(i) *B* is absolutely convex, closed and bounded;

(ii) $e \in B,\ B^2 \subseteq B \text{ and } B^* = B.$

And write $\mathcal{B}^*=\{B\in \mathcal{B}' : B=B^*\}$.

(b) If *A* is a topological *-algebra with identity *e*, as [4] then $\mathcal{B}^*$ will denote the collection of subsets B of *A* satisfying:

(i) *B* is closed and bounded;

(ii) $e \in B,\ B^2 \subseteq B \text{ and } B^* = B.$

Allan[2] defined a GB*-algebra as follows

DEFINITION 3.2 [2] A GB*-algebra is a locally convex *-algebra *A* with identity *e* such that

(i) $\mathcal{B}^*$ has a greatest member B_O,

(ii) *A* is symmetric,

(iii) $A(B)$ is a Banach *-algebra for each *B* in $\mathcal{B}^*$.

Dixon in [4] modified Alln's definition of GB*-algebras by defining it as follows:

A GB*-algebra is a topological *-algebra *A* such that

(i) $\mathcal{B}^*$ has a greatest member B_O and B_O is absolutely convex,

(ii) *A* is symmetric,

(iii) $A(B_O)$ is a Banach *-algebra.

We now come to the definition of the class of algebra so-called MGB*-algebras which is more general than the GB*-algebras (Allan's[2] definition).

DEFINITION 3.3 A MGB*-algebra is a locally convex *-algebra *A* with identity *e* such that

(i) $\mathcal{B}^*$ has a greatest member B_O and B_O is absolutely convex,

(ii) $A(B_O)$ is a Banach *-algebra.

REMARK 3.4 Every GB*-algebra (as in [2]) is a MGB*-algebra but not conversely. It is clear that each GB*-algebra is a MGB*-algebra. To show that the converse is not true in general, consider the *-algebra *A* of all complex polynomials, *p;* endowed with the topology τ of uniform convergence on compact subsets of the positive real line. It is easy to see that A_0 consists just of the constant functions and that $\mathcal{B}^*$ has a greatest member B_O, namely the set of all constant functions not exceeding unity in absolute value and $A(B_O)$ is a Banach *-algebra. Thus *A* is a MGB*-algebra. Now let $p \in A$ defined by $p(x) = x^2$ and let $q = I + pp^*$ where *I* is the identity map then $q(x) = x + x^2$. It is clear that $q \in A$ has not a bounded inverse in *A*. Thus *A* is not symmetric. Therefore *A* is not a GB*-algebra.

PROPOSITION 3.5 Let *A* be a MGB*-algebra. Then the subspace $A(B_o)$ contains all hermitian elements of A_o .

Proof. Let *h* be a hermitian element of A_o then there exists a real $\lambda \neq 0$ such that $S = \{(\lambda h)^n : n = 1,2,\dots\}$ is a bounded set. Therefore $B = \overline{S \cup \{e\}}$ is bounded (where $\overline{\ \ }$ denote the closure in *A*). Thus *B* is closed, $e \in B$, $B^2 \subseteq B$ and $B^* = B$. Hence $B \subseteq B_o$. Since $\lambda h \in B$ then $\lambda h \in B_o$. Therefore $h \in A(B_o)$. □

COROLLARY 3.6 If *A* be a symmetric MGB*-algebra, then for each $x \in A$, $(e + x^*x)^{-1} \in A(B_o)$.

Proof. Let $x \in A$. Since *A* is symmetric then $(e + x^*x)^{-1} \in A_o$. Since $(e + x^*x)^{-1}$ is hermitian element of A_o then by Proposition (3.5), $(e + x^*x)^{-1} \in A(B_o)$. □

PROPOSITION 3.7 If *A* be a symmetric MGB*-algebra, then for each $x \in A$, $x^*x(e + x^*x)^{-1} \in A(B_o)$.

Proof. Let $x \in A$. Since *A* is symmetric then by Corollary (3.6), $(e + x^*x)^{-1} \in A(B_o)$. Let $u = x^*x(e + x^*x)^{-1} = e - (e + x^*x)^{-1}$. Since both *e* and $(e + x^*x)^{-1}$ are elements of $A(B_o)$ and since $A(B_o)$ is a *-subalgebra then $e - (e + x^*x)^{-1} \in A(B_o)$.

Hence $x^*x(e + x^*x)^{-1} \in A(B_o)$. □

PROPOSITION 3.8 Let *A* be a MGB*-algebra and let *x* be any element of A_h such that $\beta(x) < +\infty$, then there exists an element $v \in A_h$ such that $v^2 = x$.

Proof. Let $x \in A_h$ such that $\beta(x) < +\infty$, then $x \in A_h \cap A_o$ and then there exists a real $\lambda \neq 0$ such that $S = \{(\lambda x)^n : n = 1,2,\dots\}$ is a bounded set . Let $B = \overline{S \cup \{e\}}$ then *B* is bounded, closed, $e \in B$, $B^2 \subseteq B$ and $B^* = B$.Hence $B \subseteq B_o$. Since $\lambda x \in B$ then $\lambda x \in B_o$.

Therefore $x \in A(B_o) \subseteq A_o$. Since $A(B_o)$ is a Banach *-algebra then by Proposition (2.19) and by [7] Lemma(4.7.2), there exists $v \in A_h$ such that $v^2 = x$. □

PROPOSITION 3.9 Let *A* be a commutative MGB*-algebra and hermitian involution. Then *A* is symmetric.

Proof . Suppose that the involution on *A* is hermitian and let $x \in A$, then by Remark(2.20) *x* has the form $x = h + ik$ where $h, k \in A_h$ then $x^*x = h^2 + k^2 \in A_h$ and then by Proposition(2.24) and Proposition(3.5), $(e + h^2)^{-1}$, $(e + k^2)^{-1}$, $h(e + h^2)^{-1}$ and $k(e + k^2)^{-1}$ are hermitian elements of A_o and then of $A(B_o)$.

Let

$$h_1 = h(e + h^2)^{-1}(e + k^2)^{-1}$$
$$k_1 = k(e + k^2)^{-1}(e + h^2)^{-1}$$

Clearly $h_1, k_1 \in A(B_O) \subseteq A_O$ and $h_1, k_1 \in A_h$. Since $h_1^2 + k_1^2 \in A(B_O) \cap A_h$ then by Proposition (3.8) there exists $v_1 \in A_h$ such that $h_1^2 + k_1^2 = v_1^2$. Then

$$h^2 + k^2 = v_1^2(e + h^2)^2(e + k^2)^2 = v^2,$$

where $v = v_1(e + h^2)(e + k^2) \in A_h$. Thus
$e + x^*x = e + h^2 + k^2 = e + v^2 \in A_h$.

Since the involution on *A* is hermitian and $e + x^*x = e + v^2 \in A_h$ then by Proposition(2.24) $e + v^2 \, and \; then \; e + x^*x$ has a bounded inverse. Therefore *A* is symmetric. □

PROPOSITION 3.10 Let *A* be a locally convex*-algebra with identity *e*. If *A* is symmetric then the envolution on *A* is hermitian.

Proof. Suppose that *A* is symmetric and let $h \in A_h$. We claim that that $\sigma_A(h)$ is real if this is not true then there is $h \in A_h$ such that $\sigma_A(h)$ contains a complex number $a + ib$, where $a, b \in \mathbb{R}$ and $b \neq 0$. Define

$$k = \frac{ah^2 + (b^2 - a^2)h}{b(b^2 + a^2)}.$$

Then $k \in A_h$ and by Theorem (2.21),

$$\sigma_A(k) = \frac{a\left(\sigma_A(h)\right)^2 + (b^2 - a^2)\sigma_A(h)}{b(b^2 + a^2)}.$$

Since $a + ib \in \sigma_A(h)$ then

$$\frac{a(a + ib)^2 + (b^2 - a^2)(a + ib)}{b(b^2 + a^2)} \in \sigma_A(k) \Rightarrow -1 \epsilon \sigma_A(k^2),$$

which implies that $e + k^2$ has no bounded inverse in *A*, which is a contradiction with the hypothesis . Therefore the involution on *A* is hermitian. □

THEOREM 3.11 Let *A* be a commutative MGB*-algebra. Then *A* is symmetric if and only if the involution is hermitian.

Proof. Since each MGB*-algebra is a locally convex*-algebra. Then the result follows from Proposition(3.9) and Proposition (3.10). □

COROLLARY 3.12 Let *A* be a commutative PL*-algebra with identity. Then *A* is symmetric if and only if the involution is hermitian.

Proof. Since each commutative PL *-algebra *A* with identity is a commutative MGB *-algebra. Then by Theorem (3.11) *A* is symmetric if and only if the involution is hermitian. □

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ملخص

في هذا البحث قدمنا صنف من الجبر الملتف المحدب موضعياً رمزنا له بالرمز *MGB كتعميم للجبر *GB المقدم من Allan. أحرزنا بعض النتائج في هذا الصنف كما حصلنا على الشروط الضرورية والكافية لأن يكون الجبر *MGB جبراً متناظراً.