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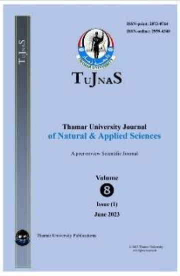

ORIGINAL ARTICLE

Partial Pre-Normality

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Abstract

The main purpose of this paper is to study a new weaker version of pre-normality called partial pre-normality, which lies between almost pre-normality (resp. quasi pre-normality) and mild pre-normality. A space   is called a partially pre-normal space if for any two disjoint closed subsets of , one of which is closed domain and the other is -closed, can be separated by two disjoint pre-open subsets. We investigate this property and present some examples to illustrate the relationships between partial pre-normality and other weaker kinds of both pre-normality and pre-regularity.

Keywords

Partially normal; Pre-normal; Almost pre-normal; Mildly pre-normal; Quasi pre-normal

1. Introduction

Throughout this paper, a space X always means a topological space on which no separation axioms are assumed, unless explicitly stated. The symbols $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{P}$ denote to the set of real, rational and irrational numbers, respectively. For a subset A of X, $X\backslash A$, $\overline{A}$ and $int(A)$ denote to the complement, the closure and the interior of A in X, respectively. A subset A of X is said to be a *regularly-open* set or an *open domain* set if it is the interior of its own closure, or equivalently if it is the interior of some closed set [1]. A complement of an open domain subset is called closed domain. A subset A of X is called a *π-closed* set if it is a finite intersection of closed domain sets [2]. A complement of a π-closed set is called *π-open*. Two sets A and B of X are said to be *separated* if there exist two disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$ [3,4,5]. A subset A of X is said to be a *pre-open* set [6] if $A \subseteq int(\overline{A})$. A subset A of X is said to be *semi open* if $A \subseteq \overline{int(A)}$ [7]. A subset A of X is called *α-open* if $A \subseteq int(\overline{int(A)})$ [8]. A space X is called a *pre-normal* space [9] if any two disjoint closed subsets A and B of X can be separated by two disjoint pre-open subsets. A space X is called an *almost pre-normal* space [8] if any two disjoint closed subsets A and B of X, one of which is closed domain, can be separated by two disjoint pre-open subsets. A space X is called a *mildly pre-normal* space [8] if any pair of disjoint closed domain subsets A and B of X, can be separated by two disjoint pre-open subsets. A space X is said to be a *partially normal* space [10] if any pair of disjoint closed subsets A and B of X, one of which is π-closed and the other is closed domain, can be separated by two disjoint open subsets. A space X is said to be a *π-pre-normal* (or πp-normal) space [11] if any pair of disjoint closed subsets A and B of X, one of which is π-closed, can be separated by two disjoint pre-open subsets. A complement of a pre-open (resp. semi open, α-open) set is called pre-closed (resp. semi closed, α-closed). An intersection of all pre-closed sets containing A is called *pre-closure* of A [12] and denoted by $p\ cl(A)$. A *pre-interior* of A denoted by $p\ int(A)$, is defined to be the union of all pre-open sets contained in A.

In this paper, we study a new weaker version of pre-normality called partial pre-normality. We show that partial pre-normality is both an additive and a topological property, and it is a hereditary property only with respect to closed domain subspaces. Some properties, examples, characterizations and preservation theorems of partial pre-normality are presented in this work.

2. Definition and Examples

First, we give the definition of partial pre-normality.

Definition 2.1 A space X is said to be a *partially pre-normal* space if for every pair of disjoint closed subsets A and B of X, one of which is π-closed and the other is closed domain, there exist disjoint pre-open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

It can be observed that every partially normal space is partially pre-normal because every open set is pre-open, and we conclude:

pre-normal ⟹ almost pre-normal ⟹ partially pre-normal ⟹ mildly pre-normal

pre-normal ⟹ quasi pre-normal ⟹ partially pre-normal ⟹ mildly pre-normal

Some counterexamples will be given in this paper to show that none of the above implications is reversible. First, we need to recall the following definition:

Definition 2.2 A space X is called a *sub-maximal* space [13,14] if every dense subset of X is an open subset.

Note that: every pre-open subset in a sub-maximal space X is an open subset. The following facts have been presented in [15] (Chapter 7).

Lemma 2.3 Let X be a space and D be a dense subset of X, then::

1. D is pre-open set in X.
2. for any subset A of X, we have $A \cup D$ is a pre-open subset.
3. for any closed subset A of X, we have $D \backslash A$ is a pre-open subset.
4. if A and B are disjoint closed subsets of X, then $(D \backslash A) \cup B$ and $(D \backslash B) \cup A$ are pre-open.
5. if X has two disjoint dense subsets, then X is pre-normal space.

Since every pre-open subset in a sub-maximal space is an open subset, we get:

Lemma 2.4 Every partially pre-normal sub-maximal space is partially normal.

Here is an example of a mildly pre-normal space but not partially pre-normal.

Example 2.5 *The irregular lattice topology* , Example 79 in [16]:

Let $X = \{(i,k): i,k \in \mathbb{Z}, i,k > 0\} \cup \{(i,0);\ i \geq 0\}$ be the subset of the integral lattice points of the plane. The irregular lattice topology on X is Urysohn, σ-compact, Lindelöf, second countable, not semi regular and has σ-locally finite base [16]. It is easy to show that the irregular lattice topology is a sub-maximal space because any dense subset of X is an open subset. Hence, every pre-open set in X is an open subset. The irregular lattice topology is a mildly normal space but not partially normal [10]. Thus, it is a mildly pre-normal space. Since X is sub-maximal non partially normal, by the Lemma 2.4 we obtain X is not partially pre-normal space.

The following is an example of a partially pre-normal space but not almost pre-normal.

Example 2.6 *The countable complement extension topology* , Example 63 in [16]:

Let $X = \mathbb{R}$ and let $\mathcal{T}_1 = \mathcal{U}$ the Euclidean topology on $\mathbb{R}$. Let $\mathcal{T}_2 = \mathcal{CC}$ the co-countable topology on $\mathbb{R}$. Define $\mathcal{T}$ to be the smallest topology generated by $\mathcal{T}_1 \cup \mathcal{T}_2$, which is called a *countable complement extension* topology on X [16]. In this space, the only open domain (closed domain, π-open, π-closed) sets in $(X, \mathcal{T})$ are those which are open domain (closed domain, π-open, π-closed) in $(\mathbb{R}, \mathcal{U})$, where $\mathcal{U}$ is the Euclidean topology on $\mathbb{R}$. It can be observed that: $(X, \mathcal{T})$ is almost regular, Lindelöf, not almost normal, not semi-regular and $\mathbb{P}$ is open. For more information about this space, see [15, 16]. Note that: in the countable complement extension topology, we have $\mathbb{P}$ is dense open subspace and any uncountable subset of $\mathbb{P}$ whose complement is countable, is also open set in X. Thus, we can easily show that any dense subset of X is an open set in X. Therefore, X is a sub-maximal space. Hence, every pre-open subset of X is an open subset. Since every almost regular Lindelöf space is quasi-normal [17], we have X is a quasi-normal space and hence partially normal. Therefore, the countable complement extension topology is a partially pre-normal space. Since X is sub-maximal non almost normal space, we obtain that X is not almost pre-normal. Hence, the countable complement extension topology is an example of a partially pre-normal space but not almost pre-normal.

Every partially normal space is partially pre-normal but the converse is not true in general as shown by the following example:

Example 2..7 Consider the product space $X = (\omega_0 + 1) \times [-1,1]$, where ω_0 is the first countable ordinal. Let $p = (\omega_0, 0) \in X$, define a topology on X by adding to the product topology of X, the basic open set of p which is the form $U_n(p) = \{p\} \cup ((\alpha, \omega_0] \times (0, \frac{1}{n})), n \in \mathbb{N}, \alpha < \omega_0$. Now, we show that X is partially pre-normal but not partially normal. To prove the space X is partially pre-normal, let A be π-closed and B be closed domain sets in X such that $A \cap B = \emptyset$. Let $G = \omega_0 + 1 \times ((-1,1) \cap \mathbb{Q})$ and $H = \omega_0 + 1 \times ((-1,1) \cap \mathbb{P})$. Then, G and H are disjoint dense subsets of X. Let $U = G \cup A$ and $V = H \cup B$. By the Lemma 2.3, we have U and V are disjoint pre-open sets in X such that $A \subseteq U$ and $B \subseteq V$. Thus, A and B can be separated by two disjoint pre-open subsets. Hence, X is partially pre-normal. Now, we show that X is not partially normal. Let $U = \{1,3,5, \dots\} \times [-1,0)$ and $V = \{0,2,4,6, \dots\} \times (0,1]$. Then, U and V are open sets in X such that $\overline{U} = (U \cup \{\omega_0\}) \times [-1,0] \setminus \{(\omega_0, 0)\}$ and $\overline{V} = (V \cup \{\omega_0\}) \times [0,1]$. Let $E = \overline{U}$ and $F = \overline{V}$. Then, E and F are disjoint closed domain sets in X. But E and F can not be separated by two disjoint open subsets. Hence, X is not mildly normal space and hence not partially normal. Therefore, the space X is an example of a partially pre-normal space but not partially normal.

The following example is a partially pre-normal space but not pre-normal.

Example 2.8 Consider the topology $\mathcal{T} = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}$ on the set $X = \{a,b,c\}$. Then, X is almost pre-normal [11]. Hence, it is partially pre-normal. But X is not pre-normal because the sets $\{b\}$ and $\{c\}$ are disjoint closed subsets of X and they can not be separated by two disjoint pre-open subsets. So, $(X, \mathcal{T})$ is an example of a partially pre-normal space but not pre-normal.

Example 2.9 *The simplified arens square topology*, Example 81 in [16], is Hausdorff, not completely Hausdorff (not Urysohn), semi regular, not regular, not normal, Lindelöf, σ-compact and with σ-locally finite base [16]. Since X is semi regular and not regular space, we get X is not almost regular. Since X is T_1 and not almost regular space, we have X is not almost normal. But X is quasi normal space [18]. Thus, it is a partially normal space and hence partially pre-normal. Since S is an open dense subspace of X, the sets $C = S \cap \mathbb{Q}$ and $D = S \cap \mathbb{P}$ are disjoint dense subsets of X. Therefore, X is a pre-normal space. Therefore, the simplified arens square topology is an example of a partially pre-normal space but not almost regular. Note that X is a pre-regular space but not regular.

3. Characterizations of Partial Pre-normality

Now, we give some characterizations of partial pre-normality.

Theorem 3.1 For a space X, the following statements are equivalent:

(a). X is partially pre-normal.

(b). for every pair of open sets U and V, one of which is open domain and the other is π-open whose union is X, there exist pre-closed subsets G and H of X such that $G \subseteq U$, $H \subseteq V$ and $G \cup H = X$.

(c). for any π-closed set A and each open domain set B such that $A \subseteq B$, there exists pre-open set U such that $A \subseteq U \subseteq p\,cl(U) \subseteq B$.

(d). for every closed domain set A and each π-open set B such that $A \subseteq B$, there exists a pre-open set U such that $A \subseteq U \subseteq p\,cl(U) \subseteq B$.

(e). for every pair of disjoint closed sets A and B of X, one of which is closed domain and the other is π-closed, there exist two pre-open subsets U and V of X such that $A \subseteq U$, $B \subseteq V$ and $p\,cl(U) \cap p\,cl(V) = \emptyset$.

Proof. $(a) \Longrightarrow (b)$. Let U be an open domain subset and V be a π-open subset of a partially pre-normal space X such that $U \cup V = X$. Then, $X\backslash U$ and $X\backslash V$ are disjoint, where $X\backslash U$ is closed domain and $X\backslash V$ is π-closed. By partial pre-normality of X, there exist disjoint pre-open subsets U_1 and V_1 of X such that $X\backslash U \subseteq U_1$ and $X\backslash V \subseteq V_1$. Let $G = X\backslash U_1$ and $H = X\backslash V_1$. Thus, G and H are pre-closed subsets of X such that $G \subseteq U$, $H \subseteq V$ and $G \cup H = X$.

$(b) \Longrightarrow (c)$. Let A be a π-closed and B be an open domain subset such that $A \subseteq B$. Then, $X\backslash A$ is π-open and B is open domain in X whose union is X. Then by (b), there exist pre-closed sets G and H such that $G \subseteq X\backslash A$, $H \subseteq B$ and $G \cup H = X$. So, $A \subseteq X\backslash G$, $X\backslash B \subseteq X\backslash H$ and $(X\backslash G) \cap (X\backslash H) = \emptyset$. Let $U = X\backslash G$ and $V = X\backslash H$. Then, U and V are disjoint pre-open sets such that $A \subseteq U \subseteq X\backslash V \subseteq B$. Since $X\backslash V$ is pre-closed, we have $p\,cl(U) \subseteq X\backslash V$. Thus, $A \subseteq U \subseteq p\,cl(U) \subseteq B$.

$(c) \Longrightarrow (d)$. Let A be closed domain and B be π-open sets in X such that $A \subseteq B$. Then, $X\backslash B \subseteq X\backslash A$, where $X\backslash B$ is π-closed and $X\backslash A$ is open domain. By (c), there exists a pre-open set V such that $X\backslash B \subseteq V \subseteq p\,cl(V) \subseteq X\backslash A$. This implies that $A \subseteq X\backslash p\,cl(V) \subseteq X\backslash V \subseteq B$. Put $U = X\backslash p\,cl(V)$. Then, U is a pre-open set in X such that $A \subseteq U \subseteq p\,cl(U) \subseteq B$.

$(d) \Longrightarrow (e)$. Let A and B be any disjoint closed sets such that A is closed domain and B is π-closed. Then, $A \subseteq X\backslash B$, where $X\backslash B$ is π-open. By (d), there exists a pre-open subset U of X such that $A \subseteq U \subseteq p\,cl(U) \subseteq X\backslash B$. Thus, we have $B \subseteq X\backslash p\,cl(U)$. Let $V = X\backslash p\,cl(U)$. Thus, V is a pre-open subset of X. Therefore, there exist two pre-open subsets U and V such that $A \subseteq U$, $B \subseteq V$ and $p\,cl(U) \cap p\,cl(V) = \emptyset$.

$(e) \Longrightarrow (a)$. It is obvious.

4. Partial Pre-normality in Subjects

The following two lemmas have been presented in [11, 15, 19].

Lemma 4.1 Let $f: X \to Y$ be a function. Then:

1. an image of a pre-open set under an open continuous function is pre-open.
2. an image of a pre-closed set under an onto, open-and-closed (clopen) continuous function is pre-closed.
3. an inverse image of a pre-open (resp. pre-closed, π-open, π-closed) set under an open continuous function is pre-open (resp. pre-closed, π-open, π-closed).

Lemma 4.2 Let M be a closed domain (resp. open, dense) subspace of X and $A \subseteq X$. If A is a pre-open (resp. pre-closed) set in X, then $A \cap M$ is a pre-open (resp. pre-closed) set in M.

Theorem 4.3 An image of a partially pre-normal space under an open continuous injective function is partially pre-normal.

Proof. Let X be a partially pre-normal space and let $f: X \longrightarrow Y$ be an open continuous injective function. We show that $f(X)$ is partially pre-normal. Let A and B be any two disjoint closed sets in $f(X)$, one of which is π-closed and the other is closed domain. Since the inverse image of a π-closed (closed domain) set under an open continuous function is π-closed (closed domain), by the Lemma 4.1, we have $f^{-1}(A)$ is π-closed in X, $f^{-1}(B)$ is closed domain in X and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. By partial pre-normality of X, there exist two pre-open subsets U and V of X such that $f^{-1}(A) \subseteq U$, $f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is an open continuous injective function, we have $A \subseteq f(U)$, $B \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. By the Lemma 4.1, we obtain $f(U)$ and $f(V)$ are disjoint pre-open sets in $f(X)$ such that $A \subseteq f(U)$ and $B \subseteq f(V)$. Hence, $f(X)$ is partially pre-normal.

From the Theorem 4.3, we obtain:

Corollary 4.4 Partial pre-normality is a topological property.

Theorem 4.5 Partial pre-normality is a hereditary property with respect to closed domain subspaces.

Proof. Let M be a closed domain subspace of a partially pre-normal space X. Let A and B be any disjoint closed sets such that A is π-closed and B is closed domain in M. Since M is a closed domain subspace of X, we have A and B are disjoint closed subsets of X, where A is π-closed and B is closed domain. By partial pre-normality of X, there exist two disjoint pre-open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. By the Lemma 4.2, we obtain $U \cap M$ and $V \cap M$ are disjoint pre-open sets in M such that $A \subseteq U \cap M$ and $B \subseteq V \cap M$. Hence, M is partially pre-normal.

Since every closed-and-open (clopen) subset is closed domain, we have:

Corollary 4.6 Partial pre-normality is a hereditary property with respect to clopen subspaces.

5. Relationships of Partial Pre-normality

In this section, we present some relationships between partial pre-normality and almost pre-regularity. First, we recall the following definitions:

Definition 5.1 A space X is called an *almost pre-regular* space if for each closed domain set F and each $x \not\in F$, there exist disjoint pre-open sets U and V such that $x \in U$ and $F \subseteq V$ [6,12].

Definition 5.2 A space X is called a *weakly pre-regular* space [8] if for each $x \in X$ and for each open domain subset U of X such that $x \in U$, there exists a pre-open subset V of X such that $x \in V \subseteq p\,cl(V) \subseteq U$.

Partial pre-normality does not imply to almost pre-regularity in general as shown by the following example.

Example 5.3 Consider the topology $\mathcal{T} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ on the set $X = \{a, b, c\}$ [11], we have X is pre-normal and hence partially pre-normal space. But X is not almost pre-regular because the closed domain $\{a, c\}$ does not contain the point b, and they do not exist two disjoint pre-open subsets containing them. So, $(X, \mathcal{T})$ is an example of a partially pre-normal space but not almost pre-regular.

Definition 5.4 A space X is called a p_1-*paracompact* space [6,20] if every pre-open cover of X has a locally finite pre-open refinement.

Clearly, every p_1-paracompact space is nearly paracompact. The following result is analogous to the Theorem 5.5 in [8].

Theorem 5.5 Every weakly pre-regular p_1-paracompact space is partially pre-normal.

Proof. Let X be a weakly pre-regular p_1-paracompact space. Since X is p_1-paracompact, it is sub-maximal and nearly paracompact, Theorem 1 in [21]. Since X is a sub-maximal and a weakly pre-regular space, it is weakly regular. In view of that fact that every weakly regular nearly paracompact space is π-normal [15], we obtain X is partially normal. Hence, X is partially pre-normal.

Since every almost pre-regular space is weakly pre-regular, we get:

Corollary 5.6 Every almost pre-regular p_1-paracompact space is partially pre-normal.

It can be observed that: partial pre-normality does not imply to almost regularity and vise versa. The simplified arens square topology, Example 2.9 is a partially normal T_1-space but not almost regular. Thus, it is partially pre-normal T_1-space but not almost regular.

Theorem 5.7 Every partially pre-normal T_1-space in which every singleton $\{x\}$ is π-closed is almost pre-regular.

Proof. It is obvious.

6. Properties of Partial Pre-normality

Now, we present the following result which is in [15].

Corollary 6.1 If $A \subseteq X = \bigoplus_{s \in S} X_s$ is π-open (resp. π-closed) in X, then $A \cap X_s$ is π-open (resp. π-closed) in X_s for each $s \in S$.

Theorem 6.2 The sum $X = \bigoplus_{s\in S} X_s,\ X_s \neq \emptyset\ \forall\ s \in S$ is partially pre-normal if and only if each X_s is partially pre-normal for each $s \in S$.

Proof. Let $X = \bigoplus_{s\in S} X_s$ be a partially pre-normal space. Since X_s is a clopen subset of X, by the Corollary 4.6 we have X_s is a partially pre-normal subspace of X for each $s \in S$. Conversely, suppose that X_s is partially pre-normal for each $s \in S$. We show that X is partially pre-normal. Let A and B be any two disjoint closed subsets of X such that A is π-closed and B is closed domain. Thus, we have $A \cap X_s$ is π-closed and $B \cap X_s$ is closed domain in X_s for each s. Since $A \cap B = \emptyset$, we have $(A \cap X_s) \cap (B \cap X_s) = \emptyset$. Since X_s is partially pre-normal for each s, there exist pre-open sets U_s and V_s in X_s such that $A \cap X_s \subseteq U_s$, $B \cap X_s \subseteq V_s$ and $U_s \cap V_s = \emptyset$. Now, since $\{X_s : s \in S\}$ is a family of pairwise disjoint topological spaces, we get $A = \cup_{s\in S} (A \cap X_s) \subseteq \cup_{s\in S} U_s = U,\ B = \cup_{s\in S} (B \cap X_s) \subseteq \cup_{s\in S} V_s = V$ and $U \cap V = (\cup_{s\in S} U_s) \cap (\cup_{s\in S} V_s) = \cup_{s\in S} (U_s \cap V_s) = \emptyset$. Thus, U and V are disjoint pre-open sets in X such that $A \subseteq U$ and $B \subseteq V$. Therefore, X is partially pre-normal.

Corollary 6.3 Partial pre-normality is an additive property.

It is well known that, a finite product of pre-open (pre-closed) sets is pre-open (pre-closed) [15]. So, we get:

Theorem 6.4 Let $(X_i, \mathcal{T}_i)$ be a space, $i = 1,2,3,\ldots,n$. If $X = \prod_{i=1}^{n} X_i$ is partially pre-normal, then $(X_i, \mathcal{T}_i)$ is partially pre-normal for each $i = 1,2,3,\ldots,n$.

Proof. Let $X = \prod_{i=1}^{n} X_i$ be partially pre-normal. Let $m \in \{1,2,3,\ldots,n\}$ be arbitrary. Let A and B be any two disjoint closed sets in X_m, where A is π-closed and B is closed domain. Let $\pi_m : \prod_{i=1}^{n} X_i \longrightarrow X_m$ be the natural projection map from X onto X_m. Now, $\pi_m^{-1}(A) = \prod_{i=1}^{n} W_i$, (where $W_i = X_i$ for each $i \neq m$) is π-closed in X, $\pi_m^{-1}(B) = \prod_{i=1}^{n} V_i$, (where $V_i = X_i$ for each $i \neq m$) is closed domain in X and $\pi_m^{-1}(A) \cap \pi_m^{-1}(B) = \emptyset$. Since X is partially pre-normal, there exist two disjoint pre-open sets U and V in X such that $\pi_m^{-1}(A) \subseteq U$ and $\pi_m^{-1}(B) \subseteq V$. Since π_m is an open onto continuous function, by the Lemma 4.1 $\pi_m(U)$ and $\pi_m(V)$ are disjoint pre-open subsets of X_m such that $A \subseteq \pi_m(U)$ and $B \subseteq \pi_m(V)$. Hence, X_m is partially pre-normal. Since m was arbitrary, we get $(X_i, \mathcal{T}_i)$ is partially pre-normal for each $i \in \{1,2,3,\ldots,n\}$.

7. Other Characterizations of Partial Pre-normality

Now, we need to recall the following definitions.

Definition 7.1 A subset A of a space X is called:

1. *g-closed* (resp. *g-open*) [21] if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open (resp. if $F \subseteq int(A)$ whenever $F \subseteq A$ and F is closed).

2. *g^*-closed* (resp. *g^*-open*) [22], if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is g-open (resp. if $F \subseteq int(A)$ whenever $F \subseteq A$ and F is g-closed).

3. *πg-closed* (resp. *πg-open*) [23] if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is π-open (resp. if $F \subseteq int(A)$ whenever $F \subseteq A$ and F is π-closed).

4. *gp-closed* (resp. *gp-open*) [24] if $p\ \mathrm{cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is open (resp. if $F \subseteq p\ \ int(A)$, whenever $F \subseteq A$ and F is closed).

5. *g^*p-closed* (resp. *g^*p-open*) [25] if $p\ cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open (resp. if $F \subseteq p\ \ int(A)$, whenever $F \subseteq A$ and F is g-closed).

6. *πgp-closed* (resp. *πgp-open*) [26] if $p\ cl(A) \subseteq U$ whenever $A \subseteq U$ and U is π-open (resp. if $F \subseteq p\ \ int(A)$, whenever $F \subseteq A$ and F is π-closed).

7. *rgp-closed* (resp. *rgp-open*) [14] if and only if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an open domain (resp. if and only if $F \subseteq p\ \ int(A)$, whenever $F \subseteq A$ and F is a closed domain).

From the Definition 7.1, we have:

closed $\Longrightarrow$ g^*-closed $\Longrightarrow$ g-closed $\Longrightarrow$ πg-closed $\Longrightarrow$ rg-closed

closed $\Longrightarrow$ pre-closed $\Longrightarrow$ g^*p-closed $\Longrightarrow$ gp-closed $\Longrightarrow$ πgp-closed $\Longrightarrow$ rgp-closed

The following theorem gives some other characterizations of partial pre-normality.

Theorem 7.2 For a space X, the following are equivalent:

(a). X is partially pre-normal.

(b). for each π-closed set A and each closed domain set B such that $A \cap B = \emptyset$, there exist disjoint g^*p-open (resp. gp-open, πgp-open, rgp-open) subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

(c) . for any π-closed set A and any open domain set B with $A \subseteq B$, there exists a g^*p-open (resp. gp-open, πgp-open, rgp-open) subset V of X such that $A \subseteq V \subseteq p\,cl(V) \subseteq B$.

(d). for any closed domain set A and each π-open set B with $A \subseteq B$, there exists a g^*p-open (resp. gp-open, πgp-open, rgp-open) subset V of X such that $A \subseteq V \subseteq p\,cl(V) \subseteq B$.

Proof. $(a) \Longrightarrow (b)$. Let X be a partially pre-normal space. Let A be π-closed and B be closed domain such that $A \cap B = \emptyset$. By partial pre-normality of X, there exist disjoint pre-open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. Thus, U and V are disjoint g^*p-open (resp. gp-open, πgp-open, rgp-open) subsets of X such that $A \subseteq U$ and $B \subseteq V$.

$(b) \Longrightarrow (c)$. Suppose (b) holds. Let A be π-closed and B be an open domain subset of X such that $A \subseteq B$. Then, $A \cap X\backslash B = \emptyset$. Thus, A and $X\backslash B$ are disjoint, where A is π-closed and $X\backslash B$ is closed domain. By (b), there exists disjoint rgp-open subsets U and V of X such that $A \subseteq U$ and $X\backslash B \subseteq V$. Therefore, we have $A \subseteq p\,int(U)$, $X\backslash B \subseteq p\,int(V)$ and $p\,int(U) \cap p\,int(V) = \emptyset$. Let $G = p\,int(U)$. Then, G is a pre-open subset of X and hence g^*p-open (resp. gp-open, πgp-open, rgp-open) such that $A \subseteq G \subseteq p\,cl(G) \subseteq B$.

$(c) \Longrightarrow (d)$. Suppose (c) holds. Let A be closed domain and B be π-open subsets of X such that $A \subseteq B$. Then, $X\backslash B \subseteq X\backslash A$, where $X\backslash B$ is π-closed and $X\backslash A$ is open domain. By (c), there exists πgp-open set U such that $X\backslash B \subseteq U \subseteq cl(U) \subseteq X\backslash A$. This implies that $X\backslash B \subseteq \mathrm{p}int(U) \subseteq \mathrm{p}int(\mathrm{p}cl(U) \subseteq \mathrm{p}cl(U) \subseteq X\backslash A$. Thus, we have $A \subseteq X\backslash \mathrm{p}cl(U) \subseteq X\backslash \mathrm{p}int(\mathrm{p}cl(U)) \subseteq X\backslash \mathrm{p}int(U) \subseteq B$. Put $V = X\backslash \mathrm{p}cl(U)$. Then, V is pre-open and hence a g^*p-open (resp. gp-open, πgp-open, rgp-open) subset of X such that $A \subseteq V \subseteq \mathrm{p}cl(V) \subseteq B$.

$(d) \Longrightarrow (a)$. Suppose (d) holds. Let A be closed domain and B be π-closed such that $A \cap B = \emptyset$. Then, we have $A \subseteq X\backslash B$ where $X\backslash B$ is π-open. By (d), there exists an rgp-open subset V of X such that $A \subseteq V \subseteq p\,cl(V) \subseteq X\backslash B$. Then, we obtain $A \subseteq p\,int(V) \subseteq V \subseteq p\,cl(V) \subseteq X\backslash B$. Let $G = p\,int(V)$ and $H = X\backslash p\,cl(V)$. Then, G and H are disjoint pre-open subsets of X such that $A \subseteq G$ and $B \subseteq H$. Hence, X is partially pre-normal.

8. Preservation Theorems on Partial Pre-normality

In this section, we present some preservation theorems of partial pre-normality. First, recall the following definitions:

Definition 8.1 A function $f: X \longrightarrow Y$ is said to be:

1. *rc-continuous* [27] if $f^{-1}(F)$ is closed domain in X for each closed domain F of Y.

2. *π-continuous* [23,28] if $f^{-1}(F)$ is π-closed in X for each closed F of Y.

3. *weakly open* [14] if for each open subset U of X, $f(U) \subseteq int(f(\overline{U}))$.

4. *R-map* [14,23] if $f^{-1}(V)$ is open domain in X for every open domain set V of Y.

5. *almost pre-irresolute* [29] if for each $x \in X$ and each pre-neighborhood V of $f(x)$ in Y, $p\,cl(f^{-1}(V))$ is a pre-neighborhood of x in X.

6. *Mp-closed* (resp. *Mp-open*) [6,12,14] if $f(U)$ is pre-closed (resp. pre-open) in Y for each pre-closed (resp. pre-open) U in X.

Lemma 8. 2, [14], If $f: X \longrightarrow Y$ is a weakly open continuous function, then f is Mp-open and R-map.

Clearly, every pre-irresolute function is almost pre-irresolute, and every π-continuous function is continuous. Now, we present the following preservation theorems of partial pre-normality.

Theorem 8.3 If $: X \longrightarrow Y$ is an Mp-open rc-continuous and almost pre-irresolute surjection function from a partially pre-normal space X onto a space Y, then Y is partially pre-normal.

Proof. Let A be a π-closed and B be an open domain subsets of Y such that $A \subseteq B$. Then by rc-continuity of f, $f^{-1}(A)$ is π-closed and $f^{-1}(B)$ is open domain subsets of X such that $f^{-1}(A) \subseteq f^{-1}(B)$. Since X is partially pre-normal, by the Theorem 3.1, there exists a pre-open subset V of X such that $f^{-1}(A) \subseteq V \subseteq p\,cl(V) \subseteq f^{-1}(B)$. Since f is Mp-open and almost pre-irresolute surjection, it follows that $f(V)$ is a pre-open subset of Y and $A \subseteq f(V) \subseteq p\,cl(f(V)) \subseteq B$. By the Theorem 3.1, we obtain Y is partially pre-normal.

Theorem 8.4 If $: X \longrightarrow Y$ is an Mp-open, open, π-continuous and almost pre-irresolute function from a partially pre-normal space X onto a space Y, then Y is partially pre-normal.

Proof. Let A be π-closed in Y and B be open domain in Y such that $A \subseteq B$. By π-continuity and openness of f, $f^{-1}(A)$ is π-closed and $f^{-1}(B)$ is open domain in X such that $f^{-1}(A) \subseteq f^{-1}(B)$. By partially pre-normality of X, there exists a pre-open set U of X such that $f^{-1}(A) \subseteq U \subseteq p\,cl(U) \subseteq f^{-1}(B)$. Since f is Mp-open almost pre-irresolute surjection, we obtain $A \subseteq$

$f(U) \subseteq p\,cl(f(U)) \subseteq B$, where $f(U)$ is pre-open in Y. By the Theorem 7.2, we get Y is partially pre-normal.

Theorem 8.5 If : $X \longrightarrow Y$ is an Mp-closed and open π-continuous function from a partially pre-normal space X onto a space Y, then Y is partially pre-normal.

Proof. Let U and V be open subsets of Y such that U is π-open and V is open domain such that $U \cup V = Y$. This implies that $f^{-1}(U) \cup f^{-1}(V) = X$. By openness π-continuity of f, $f^{-1}(U)$ is π-open in X and $f^{-1}(V)$ is open domain in X. Since X is partially pre-normal, by the Theorem 3.1 there exist pre-closed sets G and H in X such that $G \subseteq f^{-1}(U)$, $H \subseteq f^{-1}(V)$ and $G \cup H = X$. Thus, $f(G) \subseteq U$, $f(H) \subseteq V$ and $f(G) \cup f(H) = Y$. Since f is an Mp-closed function, we have $f(G)$ and $f(H)$ are pre-closed subsets of Y. By the Theorem 3.1, we obtain Y is partially pre-normal.

Theorem 8.6 If $f: X \longrightarrow Y$ is an open π-continuous, weakly open surjection and X is partially pre-normal, then Y is partially pre normal.

Proof. Let A and B be any disjoint closed subsets of Y such that A is π-closed and B is closed domain. Since f is an open π-continuous surjection, we have $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint, $f^{-1}(A)$ is π-closed and $f^{-1}(B)$ is closed domain subsets of X. Since X is partially pre-normal, there exist disjoint pre-open sets U and V in X such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since f is a weakly open continuous surjection, by the Lemma 8.2, we have f is Mp-open and R-map. Thus, $f(U)$ and $f(V)$ are disjoint pre-open sets in Y such that $A \subseteq f(U)$ and $B \subseteq f(V)$. Hence, Y is partially pre-normal.

Theorem 8.7 If $f: X \longrightarrow Y$ is an Mp-open, rc-continuous and almost pre-irresolute surjection on a partially pre-normal space X onto a space Y, then Y is partially pre-normal.

Proof. Let A be π-closed and B be open domain in Y such that $A \subseteq B$. By rc-continuity of f, we obtain $f^{-1}(A)$ is π-closed in X and $f^{-1}(B)$ is open domain in X such that $f^{-1}(A) \subseteq f^{-1}(B)$. By partial pre-normality of X, there exists a pre-open set U such that $f^{-1}(A) \subseteq U \subseteq p\,cl(U) \subseteq f^{-1}(B)$. Thus, $f(f^{-1}(A)) \subseteq f(U) \subseteq f(p\,cl(U)) \subseteq f(f^{-1}(B))$. Since f is an Mp-

open, almost pre-irresolute surjection, we obtain $A \subseteq f(U) \subseteq p\ cl(f(U)) \subseteq B$, where $f(U)$ is a pre-open subset of Y. By the Theorem 3.1, we have Y is partially pre-normal.

We can distinguish some other preservation theorems on partial pre-normality similar to those on mild pre-normality, almost pre-normality and quasi pre-normality in [8,11].

Problems: The following problems are still open in this work.

1. Is there a T_1-space X that is a partially pre-normal space but not almost pre-regular?
2. Is there an example of a partially pre-normal space but not quasi pre-normal?

9. Conclusions

A new version of pre-normality called partial pre-normality have been studied in this work. We have shown that partial pre-normality is both a topological and an additive property, and it is a hereditary property with respect to closed domain subspaces. We investigate that partial pre-normality is different from the other weaker kinds of pre-normality. Some results, properties, examples, characterizations and preservation theorems of partial pre-normality were presented.

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