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Original Article

A Group Action on an R-Module and G-Module

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Abstract

In this work, we introduce and study in the second part a new concept (the main result), which is called a group action on an R −module; in the third part, the group action on a ring R as R-module; and in the fourth part, the relation between the G-module and the group action on it. We give examples, properties, propositions, theorems, and corollaries about them.

Keywords: A Group; An Abelian Group; A Ring, An R-Module; A G-Module; A Group Action on an R-Module; A Group Action on G-Module

1. Introduction

Throughout this work, all rings are commutative, with unit only in specific places we will mention it; all modules are unitary, all group actions on an R-module are left and all groups are not necessary commutative.

In [5], the concept of a group action on a set X was introduced and studied and it was defined as follows: Let G be a group and X be a set. G is called a group action on a set X if there is a map $f: G \times X \longrightarrow X$ is defined by $f(\alpha, x) = \alpha x$, which satisfies the two axioms: $f(1_G, x) = 1_G x = x$ and $f(\alpha, x) = \alpha x, \forall \alpha \in G, \forall x \in X$. In this work, in the second part, we introduce and study the main result, which is called a left group action on an R-module. It is defined as follows: Let $(G,\cdot)$ be a group that is not necessary commutative and let R be a commutative ring. G is called a group action on an R-module M if there exist an R-module M such that $f: G \times M \longrightarrow M$ is defined by $f(\beta, x) = \beta x$ and satisfies the following axioms: $f(1_G, x) = 1_G x = x$, $f(\beta, \alpha x) = \beta(\alpha x) = (\beta \cdot \alpha)x$ and $f(\beta, x + y) = \beta(x + y) = \beta x + \beta y, \forall \beta, \alpha \in G, \forall x, y \in M$. In this case the concept of a group action on a set X.that is meaning the concept of a group action on a set X a generalization of our concept. In 2.5, we give the characterization of group action on an R-module. G is a group action on an R-module M if and only if there is $f: G \times M \longrightarrow M$ is defined by $f(\beta, x) = \beta x$ and satisfies the axioms in 2.1. We introduce and study some properties, where 2.7, is proved that G is a group action on a ring R as R-module if and only if G is a group action on an R-module M. In 2.9, if G is a group action on an R-module M, then every subgroup of G is also a group action on an R-module M. In 2.8, we clarify some examples as applications of the definition in 2.1. In 2.14, we proved that if G is a group action on R-module M, then G is a group action on every submodule of M. In 2.12, explained that if G_1 and G_2 are subgroups actions on an R-module M, then $G_1 \cap G_2$ is also group action on an R-module M. In 2.17, we generalized the result in 2.12, as in 2.13. In 2.18, we proved that if G is a group action on an R- module M_1 and an R-module M_2, hence G is group action on $M_1 + M_2$ and in 2.22, we generalized it. In 2.24, it is proved that if $R_i, i = 1,2$ are rings and R_i-modules $M_i, i = 1,2$. If G_i are group actions on R_i-modules $M_i, i = 1,2$, then $G_1 \times G_2$ is a group action on the $R_1 \times R_2$ -module $M_1 \times M_2$. In 2.25, we generalized it.

In 2.26, it is proved that if $f: G_1 \longrightarrow G_2$ is a group homomorphism. Suppose that G_2 is a group action on an R-module M and f is injective, then G_1 is a group action on an R-module M.

Finally, let R be a ring and M be an R-module. Let G be a group action on a ring R, if M is finitely generated R-module 2.31, Neotherian, 2.33, cyclic, 2.36, a simple, 3.7, free, 2.40 R-modules, then G is a group action on every one of them.

In the part three, we introduce and study the group action on a ring R as R-module such that 2.6, if R is a ring, G is called a group action on a ring R if it is a group action on a ring R as an R-module. 3.2, is proved that G is a group action on a ring R if and only if every subgroup of G is a group action on R as R-module. 3.3, is proved that if G is a group action on a ring R, then G is a group action on every ideal of a ring R. 3.4, is proved that if G is a group action on a ring R, then G is a group action on R-module R/J. In 3.5, let R be a ring and I and J are ideals of a ring R, then we have the following cases: G is a group action on $I \cap J$ as R-module, $I + J$ as R-module and $I \times J$ as R-module. 3.5, is proved that G is a group action on I and J as R- modules that is equivalent that G is a group action on $I \oplus J$ as R-module. In 3.6, we generalize the theorem in 3.5. In 3.7, and 3.8, we study if R is a PID ring and I is a maximal or a prime ideal of a ring R and G is a group action on a ring R, then G is a group action on R/I as an R-module.

In part four, we introduce and study an abelian group on the group G with the operation " $\cdot$ " which is called a G-module and is defined as: Let G be a group. A G-module consists of an abelian group M together with a group action $f: G \times M \longrightarrow M$ is defined by $f(g, m) = gm$, then $g(m_1 + m_2) = gm_1 + gm_2$, [4]. In 4.2, A G-module can is turned into a right G-module M, where $f: G \times M \longrightarrow M$ is defined by $f(g, m) = mg = g^{-1}m$. We have $g^{-1}(m_1 + m_2) = g^{-1}m_1 + g^{-1}m_2$, [4]. In 4.3, we defined G-submodule and proved that G is a group action on every G-submodule of G-module.

In 4.5, we give the characterization of G-module where M is an abelian group and G is a group with the operation " $\cdot$ ", then M is G −module if and only if G is a group action on M, and proved it. In 4.6, we give examples about G-module 4.7, is proved if $(G, \cdot)$ is a group and (A, B, C) are G-modules, then we have the following cases: G is a group action on A and C if and only if G is a group action on B and G is a group action on $A \oplus C$ if and only if G is a group action on A and C.

Finally, in 4.4, we define the homomorphism of G −module and the kernal, the image of G −homomorphism module and in 4.8, we proved that G is the group actions on $Ker(f)$ and $Im(f)$.

2. A group Action on an R-module

In this part we will explain and study the main result, which is called a group action on an R-module and is defined as:

Definition 2.1: Let $(G, \cdot)$ be a group, not necessary commutative, and let R be a commutative ring. G is called a group action on an R-module if there exists an R-module M such that $f: G \times M \longrightarrow M$ and is defined by $f(\beta, x) = \beta x$ and satisfies the following axioms:

1. $f(1_G, x) = 1_G x = x$,
2. $f(\beta, \alpha x) = \beta(\alpha x) = (\beta \cdot \alpha)x$,
3. $f(\beta, x + y) = \beta(x + y) = \beta x + \beta y, \forall \beta, \alpha \in G, \forall x, y \in M$.

Remarks 2.2: A right group action G on an R-module, if there exist an R-module M such that $f: G \times M \longrightarrow M$ is defined by $f(\beta, x) = x\beta$ and satisfies the axioms in 2.1.

Definition 2.3: *Let G be a group action on an R-module M. Then A is a subgroup action of G if and only if there is $f: A \times M \longrightarrow M$ and $f(a, m) = am$ which satisfies $\forall a, b \in A$, then $ab^{-1} \in A$ and also satisfies the axioms in 2.1.*

We introduce another formula of the definition of a group action on an R-module as:

Definition 2.4: *Let R be a ring and M be an R-module. G is called a group action on an R-module M if every subgroup of G is a group action on an R-module M.*

The following result gives the characterization of group action on an R-module.

Lemma 2.5: *Let G be a group and M be an R-module. G is a group action on an R-module M if and only if there is $f: G \times M \longrightarrow M$ is defined by $f(\beta, x) = \beta x$ and satisfies the following :*

1. $f(1_G, x) = 1_G x = x$.
2. $f(\beta, (\alpha x)) = \beta(\alpha x) = (\beta \cdot \alpha)x$.
3. $f(\beta, x + y) = \beta(x + y) = \beta x + \beta y$. $\forall \beta, \alpha \in G$ and $\forall x, y \in M$.

Proof: Assume that G is a group action on an R-module M. Then by 2.1 there exists the map f is defined from $G \times M$ to an R-module M. i.e, $f: G \times M \longrightarrow M$ by $f(\alpha, x) = \alpha x$ and satisfies : $f(1_G, x) = 1_G x = 1_G(1_R \cdot x) = (1_G \cdot 1_R)x = 1_R \cdot x = x$. $f(\alpha, \beta x) = \alpha(\beta x) = \alpha(\beta(rm)) = \alpha((\beta r)m) = \alpha(\beta r)m = (\alpha.\beta)rm = (\alpha\beta)x$ and $f(\alpha, x + y) = \alpha(x + y) = \alpha(r_1 m_1 + (\alpha r_2 m_2) = (\alpha r_1)m_1 + (\alpha r_2)m_2) = \alpha x + \alpha y$.

Conversely, Assume that there is a map $f: G \times M \longrightarrow M$ is defined by $f(\alpha, x) = \alpha x$ and satisfies the axioms in 2.1. Hence, G is a group action on an R-module M.

Definition 2.6: *Let R be a ring. G is called a group action on a ring R if it is a group action on a ring R as an R-module.*

Lemma 2.7: *Let R be a ring and M* be an *R-module. G is a group action on a ring R if and only if G is a group action on an R-module M.*

Proof: Suppose that G is a group action on the ring R. Let M be an R-module, and one can define the map $f: G \times M \longrightarrow M$ by $f(\alpha, x) = \alpha x$ and

1. $f(1_G, x) = 1_G.(rm) = (1_G.r)m = rm = x$, because G is a group action on R.
2. $f(\alpha, \beta x) = \alpha(\beta(rm)) = \alpha((\beta r)m) = (\alpha(\alpha r))m = (\alpha.\beta)rm = (\alpha.\beta)x$.
3. $f(\alpha, x + y) = \alpha(r_1 m_1 + r_2 m_2) = (\alpha r_1)m_1 + (\alpha r_2)m_2 = \alpha(r_1 m_1) + \alpha(r_2 m_2) = \alpha x + \alpha y$

$\forall x, y \in M, \forall \alpha, \beta \in G$ and $\forall r_1, r_2, r \in R$. And from 2.1 G is a group action on an R-module M. Now suppose that G is a group action on an R-module M. Then for all $\alpha \in G$ and for all $r \in R$ implies that $\alpha r \in R$. i.e; we can define the map $f: G \times R \longrightarrow R$ by $f(\alpha, r) = \alpha r$ and this map satisfies the following

- $f(1_G, r) = 1_G.r = r$,
- $f(\alpha, \beta r) = \alpha(\beta r) = (\alpha.\beta)r$,
- $f(\alpha, r_1 + r_2) = \alpha(r_1 + r_2) = \alpha r_1 + \alpha r_2$, $\forall \alpha, \beta \in G$ and $\forall r_1, r_2, r \in R$. G is a group action on a ring R.

Examples 2.8

1. The group $A = \{1, -1, i, -i\}$ is a subgroup of the field complex $\mathbb{C}$ so it is a group action on $\mathbb{C}$ that is equivalent that A is a group action on the vector space $\mathbb{V}$ on $\mathbb{C}$.
2. If M is an abelian group, then M is $\mathbb{Z}$-module and $D = \{1, -1\}$ is the group of all the inverse elements of $\mathbb{Z}$. Then D is a group action on $\mathbb{Z}$ that is equivalent to D is a group action on $\mathbb{Z}$-module M.
3. Let G be a subgroup of the field $\mathbb{R}$, then G is a group action on $\mathbb{R}$ that is equivalent that G is a group action on $\mathbb{R}$ as an $\mathbb{R}$- module.
4. Every vector space is a group action on itself.
5. The set of all matrices of the order $n \times n$ with entries from $\mathbb{R}$ is an abelian group denote it by $(M_n(\mathbb{R}), +)$ and $\{1, -1\}$ is a group action on the ring $\mathbb{Z}$ as $\mathbb{Z}$-module, hence $\{1, -1\}$ is a group action on a $\mathbb{Z}-$ module $(M_n(\mathbb{R}), +)$.

In the following we will study some results on a group G as a group action on an R-module M and its subgroups.

Proposition 2.9: *Let R be a ring and M be an R-module. Let G be a group action on M, if N is a subgroup of G, then it is a group action on M.*

Proof: Assume that G is a group action on an M. That is equivalent that from 2.7, G is a group action on an R. N is a subgroup of G that implies that 2.4, N is a group action on an R, hence from 2.7, N must be a group action on an R-module M.

Remark 2.10

1. One can easily to prove that G and $\{1_G\}$ are the trivial subgroups actions of G on an R-module M.
2. Also we can easily to prove that G is a group action on an R-trivial submodule $\{0_M\}$ of an R- module M.

Proposition 2.11: *Let R be a ring and M be an R-module. G is a group action on M if and only if every subgroup of G is a group action on M.*

Proof: Suppose that G is a group action on an R-module M, then we have G and $\{1_G\}$ are the trivial subgroups of G. And they are groups actions on M. Now let N be a proper subgroup of G, then N is a group action on an R from 2.9, N is a group action on an M.

Conversely, since every subgroup of G including G and $\{1_G\}$ are groups actions on M, hence from 2.4, G is a group action on M.

Proposition 2.12: *Let R be a ring and M be an R-module. Let G be a group action on M and G_1, G_2 are subgroups of G, then $G_1 \cap G_2$ is a group action on M.*

Proof: Suppose that G_1 and G_2 are group actions on M. $G_1 \cap G_2$ is a group action on an R-module M because $G_1 \cap G_2$ is a subgroup of G, then it is a group action on an R this implies that 2.7, $G_1 \cap G_2$ is a group action on an R-module M.

Proposition 2.13: *Let R be a ring and M be an R-module. Let G be a group action on R-module M, let $G_1, G_2, \ldots, G_n$ be subgroups of G, then $\cap_{i=1}^{n} G_i$ is a group action on an R-module M.*

Proof: For $n = 2$. Then by 2.12, $G_1 \cap G_2$ is a group action on an R-module M. Suppose that the statement is correct for n, i.e $\cap_{i=1}^{n} G_i$ is a group action on an R-module M.

We prove that is true for $n+1$. Let $G_1, G_2, \ldots, G_n, G_{n+1}$ be subgroups' actions of G, then $\cap_{i=1}^{n+1} G_i = \cap_{i=1}^{n} G_1 \cap G_{n+1}$. By 2.12, $(\cap_{i=1}^{n} G_i) \cap G_{n+1}$, is a group action on an R-module M. Then, $\cap_{i=1}^{n+1} G_i$ is a group action on an R-module M. Hence, the statement is correct for every n.

In the following, we will study some results of a group G as a group action on an R-module M and its submodules.

Proposition 2.14: *Let R be a ring and M be an R-module. Let G be a group action on an R-module M, then G is a group action on every submodule of M.*

Proof: Let N be an R-submodule of an R- module M. Since G is a group action on an R-module M, then from 2.7, G is a group action on every R-submodule N.

We will define M/N. Let N be a submodule of M. We define a set as: $M/N = \{x + N : x \in M\}$ this set with the following operations of an R-module M : $(x + N) + (y + N) = (x + y) + N$ and $\beta(x + N) = \beta x + N, \forall x, y \in M, \forall \beta \in R$. And it is easy to prove that M/N is an R-module, which is called the quotient module.

Proposition 2.15: *Let R be a ring and M be an R-module. If G is a group action on M, then G is a group action on R-module M/N.*

Proof: Suppose that G is a group action on an R-module M. Now we prove that G is a group action on R-module M/N. We define $f: G \times M/N \longrightarrow M/N$ by $f(\alpha, x + N) = \alpha(x + N) = \alpha x + N$ and we have $f(1_G, x + N) = 1_G(x + N) = 1_G x + N = x + N$, because G is a group action on M and $f(\alpha, \beta(x + N) = \alpha(\beta(x + N)) = \alpha(\beta x + N) = \alpha(\beta x) + N = (\alpha.\beta)x + N$,

Final $f(\alpha, x + N + y + N) = f(\alpha, x + y + N) = \alpha(x + y) + N = (\alpha x + \alpha y) + N = (\alpha x + N) + (\alpha y + N) = \alpha(x + N) + \alpha(y + N), \forall \alpha, \beta \in G$, $\forall x + N, y + N \in M/N$, hence from 2.5, G is a group action on R-module M/N.

Proposition 2.16: *Let R be a ring and M be an R-module. Let G be a group action on an R-module M, if N_1 and N_2 are submodules of M, then G is a group action on $N_1 \cap N_2$.*

Proof: Suppose that N_1 and N_2 are R-submodules of an R- module M. Then $N_1 \cap N_2$ is an R-submodule of an R- module M and G is a group action on an R-module M. Hence from 2.7, G is a group action on an R-submodule $N_1 \cap N_2$.

Proposition 2.17: *Let R be a ring and M be an R-module. Let $N_1, N_2, \ldots, N_n$ be R-submodules of M, then G is a group action on $\cap_{i=1}^{n} N_i$.*

Proof: Suppose that $N_1, N_2, \ldots, N_n$ are R-submodules of an R- module M, then $\cap_{i=1}^{n} N_i$ is an R-submodule of an R- module M and G is a group action on an R-module M. Hence from 2.7, G is a group action on an R-submodule $\cap_{i=1}^{n} N_i$.

One can define the sum of two submodules of an R —module M as: Let N_1, N_2 be two modules, the sum of them is defined and denoted by $N_1 + N_2 = \{x + y : x \in N_1$ and $y \in N_2\}$ and $N_1 + N_2$ is a submodule of M.

Proposition 2.18: *Let R be a ring and M be an R-module. Let G be a group action on M, if N_1 and N_2 are R-submodules of M, then G is a group action on $N_1 + N_2$.*

Proof: Let G be a group action on M, then by 2.7, G is a group action on R. And $N_1 + N_2$ is an R-submodule of M, hence by 2.7 G is a group action on $N_1 + N_2$.

Proposition 2.19: *Let R be a ring and M be an R-module. Let G be a group action on M and $N_1, N_2, \ldots, N_n$ be R-submodules of M. G is a group action on $\Sigma_{i=1}^{n} N_i$ if and only if G is a group action on R-submodules $N_i, i = 1,2,\ldots,n$.*

Proof: Let G be a group action on M, then by 2.7, G is a group action on R. Suppose that $N_1, N_2, \ldots, N_n$ are R-submodules of M, then $\Sigma_{i=1}^{n} N_i$ is an R-submodule of M that is equivalent that G is a group action on $\Sigma_{i=1}^{n} N_i$.

Conversely, let G be a group action on $\Sigma_{i=1}^{n} N_i$, then by 3.3, G is a group action on every submodule of $\Sigma_{i=1}^{n} N_i$.

Corollary 2.20: *Let R be a ring and M be an R-module. Let G be a group action on M and let $N_1, N_2, \ldots, N_n$ be submodules of M. G is a group action on $\bigoplus_{i=1}^{n} N_i$ if and only if G is a group action on R-submodules $N_i, i = 1,2,\ldots..n$.*

Proof: Since G is a group action on R-module $\{0_M\}$, hence G is a group action on R-module $\cap_{i=1}^{n} N_i = \{0_M\}$. From 2.22, G is a group action on $\bigoplus_{i=1}^{n} N_i$. And G is a group action on R-submodules $N_i, i = 1,2,\ldots,n$.

In the following theorem, we prove that G is a group action on $\Sigma_{i=1}^{n} M_i$ as R-modules.

Theorem 2.21: *Let R be a ring and $M_1, M_2, \ldots, M_n$ be R-modules. G is a group action on $\Sigma_{i=1}^{n} M_i$ if and only if G is a group action on R-modules $M_i, i = 1,2,\ldots,n$.*

Proof: $\Sigma_{i=1}^{n} M_i$ is an R-modules. Suppose that G is a group action on $\Sigma_{i=1}^{n} M_i$ R-module and from 3.3, then G is a group action on R-submodules $M_i, i = 1,2,\ldots,n$ of $\Sigma_{i=1}^{n} M_i$.

Conversely, suppose that G is a group action on R-modules $M_i, i = 1,2,\ldots..n$, then from 2.7, G is a group action on R, hence G is a group action on $\Sigma_{i=1}^{n} M_i$ an R-module.

Corollary 2.22: *Let R be a ring and $M_1, M_2, \ldots, M_n$ be R-modules. G is a group action on $\bigoplus_{i=1}^{n} M_i$ if and only if G is a group action on R-modules $M_i, i = 1,2,\ldots,n$.*

Proof: Suppose that G is a group action on R-modules $M_i, i = 1,2,\ldots,n$, then from 2.7, G is a group action on a ring R, hence G is a group action on $\bigoplus_{i=1}^{n} M_i - R$-modules and G also is a group action on $\cap_{i=1}^{n} M_i = \{0_M\}$, then G is a group action on $\bigoplus_{i=1}^{n} M_i$ as R-module.

Conversely, suppose that G is a group action on $\bigoplus_{i=1}^{n} M_i$ R-module, and from 3.3, G is a group action on every R-submodule $M_i, i = 1,2,\ldots,n$ of $\bigoplus_{i=1}^{n} M_i$ R-module.

Proposition 2.23: *Let R^* be a division ring, and $G^\star$ be a subgroup of R^*, then $G^\star$ is a group action on an R^*-module M.*

Proof: Since $G^\star$ is a subgroup of $R^\star$, that implies that $G^\star$ is a group action on $R^\star$ that is equivalent to that $G^\star$ is a group action on an $R^\star$-module M.

Proposition 2.24: *Let $R_i, i = 1,2$ be rings and R_i-modules $M_i, i = 1,2$. If G_i is a group action on R_i-modules $M_i, i = 1,2$, then $G_1 \times G_2$ is a group action on an $R_1 \times R_2$ -module $M_1 \times M_2$.*

Proof: Suppose that G_i is a group action on R_i-modules $M_i, i = 1,2$. And we can define the map $f: (G_1 \times G_2) \times (M_1 \times M_2) \longrightarrow M_1 \times M_2$ by $f((\alpha, \beta), (x, y)) = (\alpha x, \beta y)$. This map satisfies the following axioms: $f((1_{G_1}, 1_{G_2}), (x, y)) = (1_{G_1} x, 1_{G_2} y) = (x, y)$, $f((\alpha, \beta), ((\alpha_1, \beta_2)(x, y)) = f((\alpha, \beta), (\alpha_1 x, \beta_2 y)) = (\alpha(\alpha_1 x), \beta(\beta_2 y)) = ((\alpha \cdot \alpha_1)x, (\beta \cdot \beta_2)y) = ((\alpha \cdot \alpha_1), (\beta \cdot \beta_2))(x, y) = ((\alpha, \beta)(\alpha_1, \beta_2))(x,$ Finally $f((\alpha, \beta), ((x, y) + (c, d))) = ((\alpha, \beta)(x + c, y + d) = (\alpha(x + c, \beta(y + d)) = (\alpha x + \alpha c, \beta y + \beta d) = (\alpha x + \beta y) + (\alpha c + \beta d) = (\alpha, \beta)(x, y) + (\alpha, \beta)(c, d)$,

$\forall(\alpha,\beta),(\alpha_1,\beta_2) \in G_1 \times G_2$, $\forall(x,y),(c,d) \in M_1 \times M_2$. Hence $G_1 \times G_2$ is a group action on an $R_1 \times R_2$-module $M_1 \times M_2$.

Proposition 2.25: *Let R_i be rings and M_i be R_i-modules, $i = 1,2,\ldots\ldots,n$. If G_i is a group action on R_i-modules $M_i, i = 1,2,\ldots\ldots,n$,then $\Pi_{i=1}^{n} G_i$ is a group action on $\Pi_{i=1}^{n} R_i$-modules $\Pi_{i=1}^{n} M_i$.*

Proof: For $n = 2$, we have from 2.24, $\Pi_{i=1}^{2} G_i$ is a group action on $\Pi_{i=1}^{2} R_i$ -module $\Pi_{i=1}^{2} M_i$. Suppose that the statement is correct for n, i.e $\Pi_{i=1}^{n} G_i$ is a group action on an $\Pi_{i=1}^{n} R_i$-modules $\Pi_{i=1}^{n} M_i \ldots\ldots (*)$. We prove that it is true for $n+1$. Let G_i be group actions on R_i-modules $M_i, i = 1,2,\ldots\ldots,n,n+1$, then $\Pi_{i=1}^{n+1} G_i = \Pi_{i=1}^{n} G_1 \times G_{n+1}$, from 2.24, and $\Pi_{i=1}^{n} G_i$ is a group action by...(*) and G_{n+1} is a group action on an R_{n+1}-module M_{n+1}. Hence $\Pi_{i=1}^{n} G_i$ is a group action on an $\Pi_{i=1}^{n} R_i$-modules $\Pi_{i=1}^{n} M_i$ for every n.

Proposition 2.26: *Let $f: G_1 \longrightarrow G_2$ be a group homomorphism. Suppose that G_2 is a group action on an R-module M and f is injective, then G_1 is a group action on an R-module M.*

Proof: Suppose that G_2 is a group action on an R-module M, f is a group homomorphism and injective, then $G_1/Kerf \cong Imf$ and $kerf = 1_{G_1}$, then $G_1 \cong Imf$ and Imf is a subgroup of G_2, then from 2.9, Imf is a group action on an R-module M, hence G_1 is a group action on an R-module M.

Corollary 2.27: *Let R be a ring and M be an R-module. Let G^* be a group of all inverse elements in a ring R, then G^* is a group action on an R-module M.*

Proof: Since G^* is a subgroup of R. Hence G^* is a group action on R that is implies that G^* is a group action on an R-module M.

Corollary 2.28: *Let K be a field and V be K-vector space. Let Q be a subgroup of K, then Q is a group action on a K-vector space V.*

Proof: Since Q is a subgroup of K, then Q is a group action on K which is equivalent to that Q is a group action on a K-vector space V.

Theorem 2.29: *Let R be a ring and (N,M,K) be R-modules. Then we have the following cases:*

1. G is a group action on N and K if and only if G is a group action on an M.
2. G is a group action on K and N if and only if G is a group action on $N \oplus K$.

Proof: 1: Suppose that G is a group action on N and K, then G must be a group action on M because if it is not, hence by 2.7, G is not a group action on N and K and this is a contradiction.

Conversely, Suppose that G is a group action on M. If G is not a group action on N or K, then by 2.7, G is not a group action on M and this is a contradiction. Hence G is a group action on R-modules N and K.

Proof: 2: Assume that G is a group action on N and K. Apply (1) to the following short exact sequence $0 \to N \to M = N \oplus K \to K \to 0$, we get that G is a group action on M.

Conversely, suppose that G is a group action on $M = N \oplus K$. Then G is a group action on N and K, because N and K are submodules of M.

In the following, we will study if M is finitely generated R-module, then G is a group action on R-module M. And some new results for a finitely generated R-module M.

Definition 2.30 [15]: *For a ring R, an R-module M is called finitely generated (f. g for a short), for every family $\{M_i\}_{i\in I}$, (I is infinite set) of submodules of M with $M = \Sigma_{i\in I} M_i$, there is a finite subset $J \subset I$ such that $M = \Sigma_{j\in J} N_i$.*

Theorem 2.31: *Let R be a ring and M be an R-module. Let G be a group action on a ring R, if M is finitely generated R-module, then G is a group action on R-module M.*

Proof: Let G be a group action on R and let M be finitely generated R-module, then we consider $(M_i)_{i\in I}$ is a family of infinite submodules of the R-module M with $M = \Sigma_{i\in I} M_i$.

Since $M_i \subset M$ as R-modules and M is finitely generated R-module, then there exists a finite subset J of I such that $M = \Sigma_{j\in J} M_j$. Now we can define the map $f: G \times M \longrightarrow M$ by $f(\alpha,x) = \alpha x$ where $x = \Sigma_{j\in J} x_i$ and $x_i = r_i m_i$. This map satisfies the following: $f(1_G,x) = \alpha x = f(1_G, \Sigma_{j\in J} x_j) = 1_G \Sigma_{j\in J} x_j = \Sigma_{j\in J} 1_G x_j = \Sigma_{j\in J} 1_G(r_j m_j) = \Sigma_{j\in J}(1_G r_j) m_j = \Sigma_{j\in J} r_j m_j = \Sigma_{j\in J} x_j = x$, $f(\alpha,\beta x) = f(\alpha, \beta\Sigma_{j\in J} x_j) = \alpha(\beta\Sigma_{j\in J} x_j) = \alpha\Sigma_{j\in J}\beta x_j = \alpha\Sigma_{j\in J}\beta(r_j m_j) = \alpha\Sigma_{j\in J}(\beta r_j) m_j = \Sigma_{j\in J}\alpha(\beta r_j) m_j = \Sigma_{j\in J}(\alpha.\beta) r_j m_j = (\alpha.\beta)\Sigma_{j\in J} r_j m_j = (\alpha.\beta)\Sigma_{j\in J} x_j = (\alpha.\beta)x$. Finally, $f(\alpha, x+y) = f(\alpha, \Sigma_{j\in J} x_j + \Sigma_{j\in J} y_j) = \alpha(\Sigma_{j\in J} x_j + \Sigma_{j\in J} y_j) = \alpha\Sigma_{j\in J} x_j + \alpha\Sigma_{j\in J} y_j = \alpha x + \alpha y, \forall \alpha,\beta \in G, \forall x = \Sigma_{j\in J} x_j, y = \Sigma_{j\in J} y_j \in M$. Hence from 2.5, G is a group action on R-module M.

Corollary 2.32: *Let R be a ring and M be an R-module. Let G be a group action on R, if M is finitely generated R-module, then G is a group action on every submodule of R-module M.*

Proof: Let G be a group action on R and let N be a submodule of finitely generated R-module M, then N is finitely generated R-module and from 2.31, G is a group action on an R-module N.

Theorem 2.33: *Let R be a ring and M be an R-module. Let G be a group action on R, if M is Neotherian R-module, then G is a group action on M.*

Proof: Let G be a group action on R and let M be Noetherian R-module, then M is finitely generated R-module [15], and from 2.31, G is a group action on an R-module M.

Corollary 2.34: *Let R be a ring and M be an R-module. Let G be a group action on R and if M is Noetherian R-module, then G is a group action on every submodule of the R-module M.*

Proof: Let G be a group action on R and let N be a submodule of Noetherian R-module M, then N is a Noetherian R-module. and from the theorem [15], N is a finitely generated R-module and from the theorem 2.31; hence, G is a group action on an R-module N.

Definition 2.35 [15]: *An R-module M is said to be cyclic if there is an element $m_o \in M$ such that every $m \in M$ is of the form $m = rm_o$, where $r \in R$. Also m_o is called a generator of M and we can write $M = < m_o > = \{rm_o : r \in R\}$.*

Corollary 2.36: *Let R be a ring and M be an R-module. Let G be a group action on R, if M is a cyclic R-module, then G is a group action on M.*

Proof: Let G be a group action on R and let M be cyclic R-module M, then we can define the map $f: G \times M \longrightarrow M$ by $f(\alpha,m) = \alpha m$ where $m = rm_0$. This map satisfies the following : $f(1_G,m) = f(1_G, rm_0) = 1_G(rm_0) = (1_G r)m_0 = rm_0 = m$. $f(\alpha,\beta m) = f(\alpha,\beta rm_0) = \alpha(\beta rm_0) = \alpha(\beta r)m_0 = (\alpha\beta)rm_0 = (\alpha\beta)m$ .Finally $f(\alpha, m_1 + m_2) = f(\alpha, r_1 m_0 + r_2 m_0) = \alpha(r_1 m_0 + r_2 m_0) = \alpha m_1 + \alpha m_2, \forall\alpha,\beta \in G$ and $\forall m_1, m_2 \in M$. Hence from 2.5, G is a group action on R-module M.

Theorem 2.37: *Let R be a ring and M be an R-module. Let G be a group action on R, if M is a simple R-module, then G is a group action on M.*

Proof: Let G be a group action on R and let M be a submodule of a simple R-module M, then M is a cyclic R-module [15], and from the theorem 2.36, hence, G is a group action on an R-module M.

Theorem 2.38: *Let R be a ring and M be an R-module. Let G be a group action on R, if N is a maximal R-submodule of M, then G is a group action on M/N.*

Proof: Let G be a group action on R and let N be a maximal R-submodule of an R-module M, then M/N is a simple R-module that is implies that M/N is a cyclic R-module and by [15], from the theorem 2.36, hence, G is a group action on an R-module M/N.

Corollary 2.39: *Let R be a ring and M be an R-module. Let G be a group action on R, if M is a semisimple R-module, then G is a group action on M.*

Proof: Let G be a group action on R. $M = \bigoplus_{i=1}^{n} M_i$ is a semisimple R-module, then M_i are simples R-modules, then M_i are cyclic R-modules [25], and from the theorem 2.36 and 2.22, G is a group action on an R-module M.

Theorem 2.40: *Let R be a ring and M be an R-module. Let G be a group action on R and if M is a free R-module, then G is a group action on R-module M.*

Proof: Let G be a group action on R and let M be a free R-module, then M has a basis. Suppose that the basis $S = \{x_1, \dots, x_n\}$ and every element $m \in M$ can be written uniquely as $m = \Sigma r_i x_i$ for $r_l, \dots, r_n \in R$ and $x_1, x_2, \dots\dots, x_n \in S$. Now we can define the map $f: G \times M \longrightarrow M$ by $f(\alpha, m) = \alpha m$ where $m = \Sigma_{i=1}^{n} r_i x_i$. This map satisfies the following : $f(1_G, m) = \alpha m = f(1_G, \Sigma_{i=1}^{n} r_i x_i) = 1_G \Sigma_{i=1}^{n} r_i x_i = \Sigma_{i=1}^{n} 1_G (r_i x_i) = \Sigma_{i=1}^{n} ((1_G r_i) x_i) = \Sigma_{i=1}^{n} r_i m_i = m$. $f(\alpha, \beta m) = f(\alpha, \beta \Sigma_{i=1}^{n} r_i x_i) = \alpha(\beta \Sigma_{i=1}^{n} r_i x_i) = \alpha \Sigma_{i=1}^{n} (\beta r_i) x_i = \Sigma_{i=1}^{n} \alpha(\beta r_i) x_i) = \Sigma_{i=1}^{n} (\alpha.\beta) r_i x_i = (\alpha.\beta) \Sigma_{i=1}^{n} r_i x_i = (\alpha.\beta) \Sigma_{i=1}^{n} r_i x_i = (\alpha.\beta) m$. Finally $f(\alpha, x m_1 + m_2) = f(\alpha, \Sigma_{i=1}^{n} r_i x_i + \Sigma_{i=n}^{n} r_i y_i) = \alpha(\Sigma_{i=1}^{n} r_i x_i + \Sigma_{i=1}^{n} r_i y_i) = \alpha \Sigma_{j \in J} r_i x_i + \alpha \Sigma_{i=1}^{n} r_i y_i = \alpha m_1 + \alpha m_2$. $\forall \alpha, \beta \in G$ and $\forall m = \Sigma_{j \in J} r_i x_i, m_2 = \Sigma_{i=1}^{n} r_i y_i \in M$. Hence from 2.5, G is a group action on R-module M.

3. A group Action on A ring *R*.

In this section, we will study some results on a ring and its ideals. In 2.6, we defined a group action on a ring R.

Definition 3.1: *Let R be a ring. G is called a group action on a ring R if every subgroup of G is a group action on a ring R.*

Proposition 3.2: *Let R be a ring and G is a group action on a ring R if and only if every subgroup of G is a group action on a ring R as R-module.*

Proof: Suppose that G is a group action on a ring R, then we have $\{1_G\}$ is the trivial subgroup of G and it is a group action on R. Now let G_1 be a proper subgroup of G, then G_1 is a group action on an R. From 2.7, G_1 is a group action on an R as an R-module.

Conversely, since every subgroup of G including G and $\{1_G\}$ are group actions on R. Hence from 3.1, G is a group action on R as R-module.

Proposition 3.3: *Let R be a ring. Let G be a group action on a ring R, then G is a group action on every ideal of a ring R.*

Proof: Let J be an ideal of a ring R and since G is a group action on a ring R, then G is a group action on an R-submodule J, hence from 2.7, G is a group action on every R-submodule J of a ring R.

Proposition 3.4: *Let R be a ring and J be an ideal of a ring R. If G is a group action on a ring R, then G is a group action on R-module R/J.*

Proof: Assume that G is a group action on a ring R. Let J be an ideal of R, then R/J is called the quotient ring. One can easily to prove that R/J is R-module. Now we prove that G is a group action on R-module R/J. We define $f: G \times R/j \longrightarrow R/J$ by $f(\alpha, x + J) = \alpha(x + J) = \alpha x + J$ and we have $f(1_G, x + J) = 1_G(x + J) = 1_G x + J = x + J$ because G is a group action on R and $f(\alpha, \beta(x + J) = \alpha(\beta(x + J)) = \alpha(\beta x + J) = \alpha(\beta x) + J = (\alpha.\beta)x + J$ $f(\alpha, x + J + y + J) = f(\alpha, x + y + J) = \alpha(x + y) + J = (\alpha x + \alpha y) + J = \alpha(x + J) + \alpha(y + J), \forall \alpha, \beta \in G, \forall x + J, y + G \in R/J$. From 2.5, G is a group action on R-module R/J.

Theorem 3.5: *Let R be a ring and I and J be ideals of a ring R. Then we have the following cases:*

1. If G is a group action on R, then G is a group action on $I \cap J$ as R-module.
2. If G is a group action on R, then G is a group action on $I + J$ as R-module.
3. If G is a group action on R, then G is a group action on $I \times J$ as R-module.
4. If G is a group action on R. G is a group action on I and J as R modules if and only if G is a group action on $I \oplus J$ as R-module.

Proof 1, 2, and 3: Let R be a ring and I and J be ideals of a ring R, then $I \cap J$, $I + J$ and $I \times J$ are ideals of a ring R. And it is clear that $I \cap J$, $I + J$ and $I \times J$ are R-modules and G is a group action on R and from 2.5, G is a group action on $I \cap J$, $I + J$ and $I \times J$ as R-modules.

Proof 4: Suppose that G is a group action on I and J as R modules and G is a group action on R, then from 2 in 3.5, G is a group action on $I + j$ as R-module. And from 1 in 3.5, if $I \cap J = \{0\}$, then G also is a group action on it, hence G is a group action on $I \oplus j$ as R-module.

Conversely, suppose that G is a group action on $I \oplus J$. Then G is a group action on I and J. Because I and J are R-submodules of $I \oplus J$.

Corollary 3.6: *Let R be a ring and I and J be ideals of a ring R. Then we have the following cases :*

1. If G is a group action on R, then G is a group action on $\cap_{i=1}^{n} J_i$ as R-module.
2. If G is a group action on R, then G is a group action on $\Sigma_{i=1}^{n} J_i$ as R-module.
3. If G is a group action on R, then G is a group action on $\Pi_{i=1}^{n} J_i$ as R-module.
4. If G is a group action on R, then G is a group action on j_i as R-modules if and only if G is a group action on $\bigoplus_{i=1}^{n} J_i$ as R-module.

Proof 1, 2, and 3 : Let R be a ring and J_i be ideals of a ring R, then $\cap_{i=1}^{n} J_i$ is an ideal of a ring R. It is a known that $\cap_{i=1}^{n} J_i$ is R-module and G is a group action on R, hence from 2.5, G is a group action on $\cap_{i=1}^{n} J_i$. It is clear that $\Sigma_{i=1}^{n} J_i$ and $\Pi_{i=1}^{n} J_i$ are also R-modules and G is a group action on R. Hence from 2.5, G is a group action on $\Sigma_{i=1}^{n} J_i$ and $\Pi_{i=1}^{n} J_i$.

Proof 4: Suppose that G is a group action on J_i as R-modules, then from 2 in 3.5, G is a group action on ΣJ_i as R-module. And from 1 in 3.5, if $\cap J_i = \{0\}$, then G also is a group action on it, hence G is a group action on $\bigoplus_{i=1}^{n} J_i$ as R-module.

Conversely, suppose that G is a group action on $\bigoplus_{i=1}^{n} J_i$. Then G is a group action on J_i. Because J_i are R-submodules of $\bigoplus_{i=1}^{n} J_i$.

Corollary 3.7: *Let R be a ring and I be a maximal ideal of a ring R. if G is a group action on R, then G is a group action on R/I as an R-module.*

Proof: Let G be a group action on R and let I be a maximal ideal of a ring R. Then R/I is a simple R-module, which implies that R/I is a cyclic R-module [15], and from the theorem 2.36, hence G is a group action on an R-module R/I.

Corollary 3.8: *Let R be a PID ring and I be a prime ideal of a ring R, if G is a group action on R, then G is a group action on R/I as an R-module.*

Proof: Let G be a group action on R and let I be a prime ideal of a PID ring R (every ideal is a cyclic) that is equivalent I is a maximal ideal. [25], then R/I is a simple R-module that is implies that R/I is a cyclic R-module [15] and from the theorem 2.36 hence, G is a group action on an R-module R/I.

Examples 3.9

1. It is known that $(G = \{1, -1\}, \cdot)$ is a group, and it is a group action on $\mathbb{Z}$. Let $n\mathbb{Z}$, (n is a prime number) be an ideal of the ring $\mathbb{Z}$, then it is a maximal ideal because n is a prime number, then $\mathbb{Z}/n\mathbb{Z}$ is a simple $\mathbb{Z}$-module and it is a cyclic $\mathbb{Z}$- module, hence G is a group action on a $\mathbb{Z} -$ module $\mathbb{Z}/n\mathbb{Z}$.
2. It is known that $(\mathbb{Q}^*, \cdot)$ where $\mathbb{Q}$ is the field of rational numbers set

and $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ is a group and it is not a group action on $\mathbb{Z}$, then it is not a group action on $\mathbb{Z}$-module $\mathbb{Q}$ but $\mathbb{Q}^*$ it is a group action on the field $\mathbb{Q}$, hence it is a group action on $\mathbb{Z}$-module $(\mathbb{Q}, +)$.

4. A group Action on G-Module

In this part, we will introduce and study an abelian group on the group $(G, .)$ which is called the G-module, and the relation between it and the group action on it. We also study the homomorphism of G-module and some theorems and properties.

Definition 4.1 [4]: *Let G be a group. A left G-module consists of an abelian group M together with a left group action $f: G \times M \longrightarrow M$ is defined by $f(g, m) = gm$ we have $g(m_1 + m_2) = gm_1 + gm_2$.*

Remark 4.2 [4]: *A left G-module can be turned into a right G-module M, where $f: G \times M \longrightarrow M$ is defined by $(g, m) = mg = g^{-1}m$ we have $g^{-1}(m_1 + m_2) = g^{-1}m_1 + g^{-1}m_2$*

Definition 4.3 [4]: A submodule of a G-module M is a subgroup $A \subseteq M$ that is stable under the action of G, i.e $g \cdot a \in A, \forall g \in G$ and $\forall a \in A$.

Definition 4.4: *Let G be a group and let M and N be G-modules, the map $f: M \longrightarrow N$ is called a homomorphism of G-modules if and only if :*

1. $f(x + y) = f(x) + f(y)$
2. $f(\alpha x) = \alpha f(x), \forall \alpha \in G, \forall x, y \in M$.

Lemma 4.5: Let G be a group and M is an abelian group. M is G-module if and only if G is a group action on G-module M.

Proof: Suppose that M is a G-module, then by 4.1, M is an abelian group, and there is a map $f: G \times M \longrightarrow M$ is defined by $f(\alpha, x) = \alpha x$ and M is G-module, then satisfies the following : $f(1_G, x) = 1_G x = x, f(\alpha, \beta x) = \alpha(\beta x) = (\alpha.\beta)x$. Finally $f(\alpha, x + y) = \alpha(x + y) = \alpha x + \alpha y, \forall \alpha, \beta \in G$ and $\forall x, y \in M$, hence G is a group action on G-module M.

Conversely, assume that G is a group action on G-module M, then satisfies the axioms in 3.4 and we have $f(\alpha, x + y) = \alpha(x + y) = \alpha x + \alpha y, \forall \alpha, \beta \in G, \forall x, y \in M$. And from 4.1, M is a G-module.

We study the following two important examples:

Examples 4.6

1. It is known that $(\mathbb{Q}, +, \cdot)$ is the field of rational numbers set and $(\mathbb{Q}^*, \cdot) = \mathbb{Q} - \{0\}$ is a group and it is a group action on the abelian group $(\mathbb{Q}, +)$ as $\mathbb{Q}$-module. then $(\mathbb{Q}, +)$ is $\mathbb{Q}^*$-module.
2. The set of all matrices of the order 2×2 with entries from $\mathbb{R}$ is an abelian group denoted by $(M_2(\mathbb{R}), +)$ and $(\mathbb{Q}^*, \cdot)$ is a group action on $M_2(\mathbb{R})$ and hence $M_2(\mathbb{R})$ is $\mathbb{Q}^*$-module.

Theorem 4.7: *Let $(G, \cdot)$ be a group and (A, B, C) be G-modules, then we have the following cases :*

1. G is a group action on A and C if and only if G is a group action on an B.
2. G is a group action on $A \oplus C$ if and only if G is a group action on A and C.

Poof 1: Assume that G is a group action on A and C, then G must be a group action on B because if it is not, then by 4.5, B is not G-module, and this is a contradiction.

Conversely, suppose that G is a group action on B. G must be a group action on A and C. Because if it is not a group action on A or C, then by 4.5, A or C is not G- module and G is not a group action on B this a contradiction. Hence G is a group action on A and C.

Poof 2: Suppose that G is a group action on A and C. Apply (1) to the following short exact sequence $0 \longrightarrow A \longrightarrow B = A \oplus C \longrightarrow C \longrightarrow 0$, we get that G is a group action on B.

Conversely, suppose that G is a group action on $B = A \oplus C$, then G is a group action on A and C because A and C are G-submodules of G-module B.

In the following theorem we prove that G is a group action on $Kerf$ and Imf.

Theorem 4.8: Let M, N be G -modules and $f: M \longrightarrow N$ be a homomorphism of G-modules. If G is a group action on M and N, then G is a group action on $Kerf$ and Imf.

Proof: Suppose that G is a group action on a G-module M and $Kerf$ is a G-submodule of M. Hence from 2.14, G is a group action on $Kerf$ as a G-submodule of M. And also G is a group action on a G-module N and Imf is a G-submodule of G-module N. Then by 2.14, G is a group action on Imf.

Data Availability

The datasets used and analyzed during the current study are available from the corresponding author upon reasonable request.

Conflict of Interest

The authors declare no conflict of interest.

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