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Original Article

A Complex Al-Zughair Transform

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Abstract

This paper aims to introduce a new transform known as the complex Al-Zughair transform, which has been presented in previous research. A complex Al-Zughair transform is useful for finding the transform of new functions.

Keywords: Integral transform; Al-Zughair transform; Complex Al-Zughair transform; Differential equation

1. Introduction

In 2008, Mohammed and Kathem discovered Al-Tememe transform [1] which has been used to solve Euler's Equation. In 2015, Al-Tememe transform has been used to solve a special type of differential equation [2]. A new transform known as the “Al-Zughair transform” was discovered [3], which was taken on to solve special types of differential equations [4]. The transform has the following formula:

$$Z\big(f(\varkappa)\big) = \int_1^e \frac{(\ln(\varkappa))^{\text{Ᵽ}}}{\varkappa} f(\varkappa) d\varkappa \quad \ldots\ldots\ldots\ldots \ (1)$$

Al-Zughair transformation was used to define some basic concepts such as differentiation, integration, solving linear systems of ordinary and partial equations, and convolution theory.

Definition (1.1) [1]: Suppose that f is a function defined on the interval (a, b). The integral transform for f whose symbol $F(\text{Ᵽ})$ such as:

$$F(\text{Ᵽ}) = \int_a^b k(\text{Ᵽ}, \varkappa) f(\varkappa) d\varkappa,$$

In which k is a function of two factors, Ᵽ and $\varkappa$ and it's called the kernel of the transform and a, b ∈ IR or $\pm\infty$, such that the above integral converges.

Definition (1.2) [1]: Suppose that $f(x)$, is a function defined in [1, e] Al-Zughair transform is characterized by the integral:

$$Z\big(f(\varkappa)\big) = \int_1^e \frac{(\ln(\varkappa))^{\text{Ᵽ}}}{\varkappa} f(\varkappa)\, d\varkappa \equiv F(\text{Ᵽ})$$

Such that this integral converges and Ᵽ > -1.

Definition (1.3): Suppose that $f(\varkappa)$, is a function defined in [1, e]. The complex Al-Zughair transform is characterized by the integral:

$$Z^c\big(f(\varkappa)\big) = \int_1^e \frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa} f(\varkappa)\, d\varkappa \equiv F(\text{Ᵽ}i) \ldots (2)$$

Such that this integral converges and $P \in R$, Ᵽ > -1, and $\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}$ is the kernel of this transform, $i = \sqrt{-1}$.

Property (1.4):

A complex Al-Zughair transform is linear.

$$Z^c\big(Af(\varkappa) \pm Bg(\varkappa)\big) = \int_1^e \Big(Af(\varkappa) \pm Bg(\varkappa)\Big) \frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa} d\varkappa \,; A \text{ and } B \text{ real numbers}$$

$$= \int_1^e Af(\varkappa) \frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa} d\varkappa \pm \int_1^e Bg(\varkappa) \frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa} dx$$

$$= A\int_1^e f(\varkappa) \frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa} d\varkappa \pm B\int_1^e g(\varkappa) \frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa} d\varkappa$$

$$= AZ^c\big(f(\varkappa)\big) \pm BZ^c\big(g(\varkappa)\big)$$

1.2 A complex Al-Zughair transform of some functions

1) $Z^c(1) = \frac{1}{1+\text{Ᵽ}^2} - \frac{\text{Ᵽ}}{1+\text{Ᵽ}^2} i$

Proof:

$$Z^c(1) = \int_1^e \frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa} d\varkappa = \frac{(\ln(\varkappa))^{\text{Ᵽ}i+1}}{\text{Ᵽ}i+1}\bigg|_1^e = \frac{(\ln(e))^{1+\text{Ᵽ}i}}{1+\text{Ᵽ}i} - \frac{(\ln(1))^{1+\text{Ᵽ}i}}{1+\text{Ᵽ}i}$$

$$=\frac{1}{1+Ᵽi}\times\frac{-Ᵽi+1}{-Ᵽi+1}=\frac{-Ᵽi+1}{1+Ᵽ^2}=\frac{1}{Ᵽ^2+1}-\frac{Ᵽ}{Ᵽ^2+1}i$$

2) $Z^c(k)=\frac{k}{Ᵽ^{2+1}}-\frac{kᵽ}{1+Ᵽ^2}i$ where $k\in R$

Proof:

$$Z^c(k)=\int_1^e k\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}dx=k\frac{(\ln(\varkappa))^{Ᵽi+1}}{Ᵽi+1}\Bigg|_1^e$$
$$=k\frac{(\ln(e))^{Ᵽi+1}}{Ᵽi+1}-k\frac{(\ln(1))^{Ᵽi+1}}{Ᵽi+1}$$
$$=\frac{k}{1+iᵽ}\times\frac{-Ᵽi+1}{-Ᵽi+1}=k\frac{(1-Ᵽi)}{Ᵽ^2+1}=\left(k/(1+Ᵽ^2)\right)-\frac{kᵽ}{1+Ᵽ^2}i$$

3) $Z^c(\ln(\varkappa))=\frac{2}{4+Ᵽ^2}-\frac{Ᵽi}{4+Ᵽ^2}$

Proof:

$$Z^c(\ln\varkappa)=\int_1^e\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}\ln\varkappa\,d\varkappa=\int_1^e\frac{(\ln(\varkappa))^{Ᵽi+1}}{\varkappa}d\varkappa==\frac{(\ln(\varkappa))^{2+Ᵽi}}{2+Ᵽi}\Bigg|_1^e$$
$$=\frac{(\ln(e))^{2+Ᵽi}}{2+Ᵽi}-\frac{(\ln(1))^{2+Ᵽi}}{2+Ᵽi}$$
$$=\frac{1}{2+Ᵽi}\times\frac{2-Ᵽi}{2-Ᵽi}=\frac{2-Ᵽi}{4+Ᵽ^2}=\frac{2}{4+Ᵽ^2}-\frac{Ᵽ}{4+Ᵽ^2}i.$$

4) $Z^c((\ln(\varkappa)^ȵ)=\frac{(ȵ+1)}{(ȵ+1)^2+Ᵽ^2}-\frac{Ᵽi}{(ȵ+1)^2+Ᵽ^2}$; $ȵ$ *integer number*

Proof:

$$Z^c((\ln(\varkappa))^ȵ)=\int_1^e\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}(\ln(\varkappa))^ȵ\,d\varkappa=\int_1^e\frac{(\ln(\varkappa))^{Ᵽi+ȵ}}{\varkappa}d\varkappa$$
$$=\frac{(\ln(\varkappa))^{Ᵽi+(ȵ+1)}}{Ᵽi+(ȵ+1)}\Bigg|_1^e=$$
$$=\frac{(\ln(e))^{Ᵽi+(ȵ+1)}}{Ᵽi+(1+ȵ)}-\frac{(\ln(1))^{Ᵽi+(ȵ+1)}}{Ᵽi+(1+ȵ)}$$
$$=\frac{1}{Ᵽi+(ȵ+1)}\times\frac{-Ᵽi+(ȵ+1)}{-Ᵽi+(1+ȵ)}=\frac{(ȵ+1)-Ᵽi}{(ȵ+1)^2+Ᵽ^2}$$
$$=\frac{ȵ+1}{(1+ȵ)^2+Ᵽ^2}-\frac{Ᵽ}{(1+ȵ)^2+Ᵽ^2}i$$

5) $Z^c(\ln(\ln(\varkappa))=-\frac{(1-Ᵽi)^2}{(1+Ᵽ^2)^2}$

Proof:

$$Z^c(\ln(\ln(\varkappa))=\int_1^e\ln(\ln(\varkappa))\frac{(\ln(x))^{Ᵽi}}{x}d\varkappa$$

Let $u=\ln(\ln(x))\Rightarrow du=\frac{1}{\ln(x)}.\frac{1}{x}d\varkappa,\ dv=\frac{(\ln(x))^{Ᵽi}}{\varkappa}\Rightarrow v=\frac{(\ln(x))^{Ᵽi+1}}{1+Ᵽi}$

$$=\ln(\ln(x))\frac{(\ln(\varkappa))^{Ᵽi+1}}{1+Ᵽi}\Bigg|_1^e-\int_1^e\frac{(\ln(\varkappa))^{1+Ᵽi}}{1+Ᵽi}.\frac{1}{\ln(x)}.\frac{1}{x}d\varkappa$$
$$=-\int_1^e\frac{(\ln(x))^{Ᵽi}}{(Ᵽi+1)}.\frac{1}{\varkappa}d\varkappa=-\frac{(\ln(\varkappa))^{Ᵽi+1}}{(1+Ᵽi)^2}\Bigg|_1^e=\frac{-1}{(1+Ᵽi)^2}\times\frac{(1-Ᵽi)^2}{(1-Ᵽi)^2}$$
$$=-\frac{(1-Ᵽi)^2}{(1+Ᵽ^2)^2}$$

6) $Z^c((\ln(\ln(\varkappa))^ȵ)=(-1)^ȵȵ!\frac{(1-Ᵽi)^{ȵ+1}}{(1+Ᵽ^2)^{ȵ+1}}$ where $ȵ$ integer number.

Proof:

$$Z^c((\ln(\ln(\varkappa)))^ȵ)=\int_1^e(\ln(\ln(\varkappa))^ȵ\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}d\varkappa$$

Let $u=(\ln(\ln(\varkappa)))^ȵ\Rightarrow du=ȵ(\ln(\ln(\varkappa)))^{ȵ-1}\frac{1}{l'n(x)}.\frac{1}{x}d\varkappa$, $dv=\frac{(\ln(\varkappa))^{Ᵽi}}{x}\Rightarrow$ $v=\frac{(l'n(x))^{Ᵽi+1}}{1+Ᵽi}$

$$=(\ln(\ln(x)))^ȵ\frac{(\ln(x))^{1+Ᵽi}}{1+Ᵽi}\Bigg|_1^e-\int_1^e\frac{(l'n(x))^{1+Ᵽi}}{Ᵽi+1}.\frac{ȵ(l'n(\ln(x)))^{ȵ-1}}{\ln(\varkappa)}.\frac{1}{\varkappa}d\varkappa$$
$$=-\int_1^e\frac{(\ln(\varkappa))^{Ᵽi}}{(Ᵽi+1)}.\frac{ȵ(\ln(\ln(\varkappa)))^{ȵ-1}}{\varkappa}d\varkappa$$

$u=(\ln(\ln(\varkappa)))^{ȵ-1}\Rightarrow du=(ȵ-1)(\ln(\ln(\varkappa)))^{ȵ-2}\frac{1}{\varkappa ln(\varkappa)}d\varkappa$

$dv=\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}\ \Rightarrow\ v=\frac{(\ln(\varkappa))^{Ᵽi+1}}{Ᵽi+1}$

$$=\frac{-ȵ}{(Ᵽi+1)}\left[(\ln(\ln(\varkappa)))^{ȵ-1}\frac{(\ln(\varkappa))^{1+Ᵽi}}{1+Ᵽi}\Bigg|_1^e-\int_1^e\frac{(\ln(\varkappa))^{Ᵽi+1}}{Ᵽi+1}.\frac{(ȵ-1)(\ln(\ln(\varkappa)))^{ȵ-2}}{\ln(\varkappa)}.\frac{1}{\varkappa ln\varkappa}dx\right]$$
$$=\frac{ȵ(ȵ-1)}{(Ᵽi+1)^2}\int_1^e\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}(\ln(\ln(\varkappa)))^{ȵ-2}d\varkappa$$

$u=(\ln(\ln(\varkappa)))^{ȵ-2}\ \Rightarrow\ du=(ȵ-2)(\ln(\ln(\varkappa)))^{ȵ-3}\frac{1}{\varkappa ln\varkappa}d\varkappa$

$dv=\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}\ \Rightarrow\ v=\frac{(\ln(\varkappa))^{Ᵽi+1}}{Ᵽi+1}$

$$=\frac{ȵ(ȵ-1)}{(Ᵽi+1)^2}\left[(\ln(\ln(\varkappa)))^{ȵ-2}\frac{(\ln(\varkappa))^{1+Ᵽi}}{1+Ᵽi}\Bigg|_1^e-\int_1^e\frac{(\ln(\varkappa))^{1+Ᵽi}}{1+Ᵽi}.(ȵ-2)(\ln(\ln(\varkappa)))^{ȵ-3}.\frac{1}{\varkappa ln\varkappa}d\varkappa\right]$$
$$=-\frac{ȵ(ȵ-1)(ȵ-2)}{(1+Ᵽi)^3}\int_1^e\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}.(\ln(\ln(\varkappa)))^{ȵ-3}d\varkappa$$

$\vdots$

$$=\frac{(-1)^ȵȵ!}{(1+Ᵽi)^{ȵ+1}}.\frac{(1-Ᵽi)^{ȵ+1}}{(1-Ᵽi)^{ȵ+1}}=\frac{(-1)^ȵȵ!\,(1-Ᵽi)^{ȵ+1}}{(1+Ᵽi)^{ȵ+1}}$$

We can also prove by mathematical induction that:

If $n=2\ \Rightarrow\ Z^c((\ln(\ln(\varkappa)))^2)=\int_1^e(\ln(\ln(\varkappa)))^2\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}d\varkappa$

$u=(\ln(\ln(\varkappa)))^2\ \Rightarrow du=2(\ln(\ln(\varkappa))).\frac{1}{\varkappa\ln\varkappa}d\varkappa$

$dv=\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}\ \Rightarrow\ v=\frac{(\ln(\varkappa))^{Ᵽi+1}}{(Ᵽi+1)}$

$$=\left[(\ln(\ln(\varkappa)))^2\frac{(\ln(\varkappa))^{Ᵽi+1}}{Ᵽi+1}\Bigg|_1^e-\int_1^e\frac{(\ln(\varkappa))^{Ᵽi+1}}{Ᵽi+1}2(\ln(\ln(\varkappa)))^1.\frac{1}{\varkappa ln\varkappa}d\varkappa\right]$$
$$=\frac{-2}{(Ᵽi+1)}Z^c((\ln(\ln(\varkappa)))^1)=\frac{-2}{(Ᵽi+1)}.\frac{-1}{(Ᵽi+1)^2}=\frac{2}{(1+Ᵽi)^3}.\frac{(1-Ᵽi)^3}{(1-Ᵽi)^3}$$
$$=\frac{2(1-Ᵽi)^3}{(1+Ᵽ^2)^3}$$

If $ȵ=3\ \Rightarrow\ Z^c((\ln(\ln(\varkappa)))^3)=\int_1^e(\ln(\ln(\varkappa)))^3\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}d\varkappa$

$u=(\ln(\ln(\varkappa)))^3\ \Rightarrow\ du=3(\ln(\ln(\varkappa)))^2\frac{1}{\varkappa ln\varkappa}d\varkappa$

$dv=\frac{(\ln(\varkappa))^{Ᵽi}}{\varkappa}\ \Rightarrow\ v=\frac{(\ln(\varkappa))^{Ᵽi+1}}{Ᵽi+1}$

$$=\left[(\ln(\ln(\varkappa)))^3\frac{(\ln(\varkappa))^{Ᵽi+1}}{Ᵽi+1}\Bigg|_1^e-\int_1^e\frac{(\ln(\varkappa))^{Ᵽi+1}}{Ᵽi+1}3(\ln(\ln(\varkappa)))^2.\frac{1}{\varkappa ln\varkappa}d\varkappa\right]$$
$$=\frac{-3}{(Ᵽi+1)}Z^c((\ln(\ln(\varkappa)))^2)=\frac{-3}{(Ᵽi+1)}.\frac{2}{(1+Ᵽi)^3}=-\frac{6}{(1+Ᵽi)^4}$$
$$=\frac{-6}{(1+Ᵽi)^4}.\frac{(1-Ᵽi)^4}{(1-Ᵽi)^4}=\frac{-6(1-Ᵽi)^4}{(1+Ᵽ^2)^4}$$

$\vdots\left(\left((())\right)\right)$

$$Z^{c}((ln(ln(\varkappa)))^{\eta}) = (-1)^{\eta}\eta!\frac{(1-\text{Ᵽ}i)^{\eta+1}}{(1+\text{Ᵽ}^2)^{\eta+1}}$$

7) $Z^{c}\left(\sin\left(aln(ln\,(\varkappa))\right)\right) = \int_1^e \sin\left(aln(ln\,(\varkappa))\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$ where $a \in R$.

$$= \int_1^e \left(\frac{e^{aln(ln\varkappa))i} - e^{-aln(ln\varkappa))i}}{2i}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$= \int_1^e \left(\frac{e^{\ln\left((ln\varkappa)^{ai}\right)} - e^{\ln\left((ln\varkappa)^{-ai}\right)}}{2i}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$= \frac{1}{2i}\left(Z^{c}\left((ln\varkappa)^{ai}\right) - Z^{c}\left((ln\varkappa)^{-ai}\right)\right) = \frac{1}{2i}\left(\frac{1}{\text{Ᵽ}i+(1+ia)} - \frac{1}{\text{Ᵽ}i+(-ai+1)}\right)$$

$$= \frac{1}{2i}\left(\frac{\text{Ᵽ}i - ai + 1 - \text{Ᵽ}i - ai - 1}{(i(\text{Ᵽ}+a)+1)(i(\text{Ᵽ}-a)+1)}\right) = \frac{1}{2i}\frac{-2ai}{(i(\text{Ᵽ}+a)+1)(i(\text{Ᵽ}-a)+1)}$$

$$= \frac{-a}{(i(\text{Ᵽ}+a)+1)(i(\text{Ᵽ}-a)+1)} = \frac{-a}{a^2-(\text{Ᵽ}^2-2\text{Ᵽ}i-1)}$$

$$= \frac{-a}{a^2-(\text{Ᵽ}-i)^2}\times\frac{a^2-(i+\text{Ᵽ})^2}{a^2-(i+\text{Ᵽ})^2}$$

$$= \frac{-a^3+a\text{Ᵽ}^2-a}{a^4-2a^2(\text{Ᵽ}^2-1)+(\text{Ᵽ}^2+1)^2} + \frac{2a\text{Ᵽ}i}{a^4-2a^2(\text{Ᵽ}^2-1)+(\text{Ᵽ}^2+1)^2}$$

8) $Z^{c}\left(\cos\left(aln(ln\varkappa)\right)\right) = \int_1^e \cos\left(aln(ln\varkappa)\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$

$$= \int_1^e \left(\frac{e^{aln(ln\varkappa))i} + e^{-aln(ln\varkappa))i}}{2}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$= \int_1^e \left(\frac{e^{\ln\left((\ln(\varkappa))^{ai}\right)} + e^{\ln\left((\ln(\varkappa))^{-ai}\right)}}{2}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$= \frac{1}{2}\left(Z^{c}\left((\ln(\varkappa))^{ai}\right) + Z^{c}\left((\ln(\varkappa))^{-ai}\right)\right)$$

$$= \frac{1}{2}\left(\frac{(ia+1)-\text{Ᵽ}i}{\text{Ᵽ}^2+(ia+1)^2} + \frac{(-ia+1)-\text{Ᵽ}i}{\text{Ᵽ}^2+(1-ai)^2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{(\text{Ᵽ}i+(ai+1))} + \frac{1}{(\text{Ᵽ}i+(-ai+1))}\right)$$

$$= \frac{1}{2}\frac{2(1+\text{Ᵽ}i)}{\left((1+\text{Ᵽ}i)+ai\right)\left((1+\text{Ᵽ}i)-ai\right)}$$

$$= \frac{(\text{Ᵽ}i+1)}{((\text{Ᵽ}i+1)^2+a^2)}\times\frac{((-\text{Ᵽ}i+1)^2+a^2)}{((-\text{Ᵽ}i+1)^2+a^2)}$$

$$= \frac{(1+\text{Ᵽ}^2)+a^2}{a^4-2a^2(-1+\text{Ᵽ}^2)+(1+\text{Ᵽ}^2)^2} - \frac{\text{Ᵽ}\left((\text{Ᵽ}^2+1)-a^2\right)i}{a^4-2a^2(\text{Ᵽ}^2-1)+(\text{Ᵽ}^2+1)^2}$$

9) $Z^{c}\left(\sinh\left(aln(ln\varkappa)\right)\right) = \int_1^e \sinh\left(aln(ln\varkappa)\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$

$$= \int_1^e \left(\frac{e^{aln(\ln(\varkappa)))} - e^{-aln(\ln(\varkappa)))}}{2}\right)\frac{(ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$= \int_1^e \left(\frac{e^{\ln((\ln(\varkappa))^{a})} - e^{\ln((\ln(\varkappa))^{-a})}}{2}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$= \frac{1}{2}\left(Z^{c}((\ln(\varkappa))^{a}) - Z^{c}((\ln(\varkappa))^{-a})\right) = \frac{1}{2}\left(\frac{1}{\text{Ᵽ}i+(a+1)} - \frac{1}{\text{Ᵽ}i+(1-a)}\right)$$

$$= \frac{1}{2}\left(\frac{\text{Ᵽ}i-a+1-\text{Ᵽ}i-a-1}{((\text{Ᵽ}i+1)+a)((\text{Ᵽ}i+1)-a)}\right) = \frac{1}{2}\frac{-2a}{((\text{Ᵽ}i+1)+a)((\text{Ᵽ}i+1)-a)}$$

$$= \frac{-a}{(\text{Ᵽ}i+1)^2-a^2} = \frac{-a}{(\text{Ᵽ}i+1)^2-a^2}\times\frac{(-\text{Ᵽ}i+1)^2-a^2}{(-\text{Ᵽ}i+1)^2-a^2}$$

$$= \frac{-a(1-\text{Ᵽ}^2-a^2)}{(1+\text{Ᵽ}^2)^2-2a^2(1-\text{Ᵽ}^2)+a^4} + \frac{2a\text{Ᵽ}i}{(1+\text{Ᵽ}^2)^2-2a^2(1-\text{Ᵽ}^2)+a^4}$$

10) $cosh\left(aln(\ln(\varkappa))\right) = \int_1^e \cosh\left(aln(\ln(\varkappa))\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$

$$= \int_1^e \left(\frac{e^{aln(\ln(\varkappa)))} + e^{-aln(\ln(\varkappa)))}}{2}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$= \int_1^e \left(\frac{e^{\ln((\ln(\varkappa))^{a})} + e^{\ln((\ln(\varkappa))^{-a})}}{2}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$= \frac{1}{2}\left(Z^{c}((\ln(\varkappa))^{a}) + Z^{c}((\ln(\varkappa))^{-a})\right) = \frac{1}{2}\left(\frac{1}{\text{Ᵽ}i+(a+1)} + \frac{1}{\text{Ᵽ}i+(1-a)}\right)$$

$$= \frac{1}{2}\left(\frac{\text{Ᵽ}i-ai+1-\text{Ᵽ}i-ai-1}{(\text{Ᵽ}i+(a+1))(\text{Ᵽ}i+(-a+1))}\right) = (1/2)\frac{2(\text{Ᵽ}i+1)}{((\text{Ᵽ}i+1)+a)((\text{Ᵽ}i+1)-a)}$$

$$= \frac{(1+\text{Ᵽ}i)}{((1+\text{Ᵽ}i)^2-a^2)} = \frac{(\text{Ᵽ}i+1)}{((\text{Ᵽ}i+1)^2-a^2)}\times\frac{((-\text{Ᵽ}i+1)^2-a^2)}{((-\text{Ᵽ}i+1)^2-a^2)}$$

$$= \frac{\text{Ᵽ}^2-a^2+1}{a^4+2a^2(\text{Ᵽ}^2-1)+(\text{Ᵽ}^2+1)^2} - \frac{(\text{Ᵽ}+\text{Ᵽ}^3+a^2\text{Ᵽ})i}{a^4+2a^2(-1+\text{Ᵽ}^2)+(1+\text{Ᵽ}^2)^2}$$

11) $Z^{c}\left((\ln(\varkappa))^{m}\cos\left(aln(\ln(\varkappa))\right)\right) = \int_1^e (\ln(\varkappa))^{m}\cos\left(aln(\ln(\varkappa))\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$

$$= \int_1^e \left(\frac{e^{aln(\ln(\varkappa)))i} + e^{-aln(\ln(\varkappa)))i}}{2}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}(\ln\varkappa)^{m}d\varkappa$$

$$= \int_1^e \left(\frac{e^{\ln\left((\ln(\varkappa))^{ai}\right)} + e^{\ln\left((\ln(\varkappa))^{-ai}\right)}}{2}\right)(\ln(\varkappa))^{m}\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$= \frac{1}{2}\left(Z^{c}\left((\ln(\varkappa))^{m+ai}\right) + Z^{c}\left((\ln(\varkappa))^{m-ai}\right)\right) = (1/2)\left(\frac{1}{\text{Ᵽ}i+(m+ai+1)} + \frac{1}{\text{Ᵽ}i+(m-ia+1)}\right)$$

$$= \frac{1}{2}\left(\frac{2\text{Ᵽ}i+2m+2}{-\text{Ᵽ}^2+a^2+2m\text{Ᵽ}i+2\text{Ᵽ}i+m^2+2m+1}\right)$$

$$= \frac{\left(\text{Ᵽ}i+(m+1)\right)}{(i^2\text{Ᵽ}^2+2\text{Ᵽ}i(m+1)+(m+1)^2+a^2)}$$

$$= \frac{\left(\text{Ᵽ}i+(m+1)\right)}{\left(\left(\text{Ᵽ}i+(m+1)\right)^2+a^2\right)}\times\frac{\left(\left(-\text{Ᵽ}i+(m+1)\right)^2+a^2\right)}{\left(\left(-\text{Ᵽ}i+(m+1)\right)^2+a^2\right)}$$

$$= \frac{\left(\text{Ᵽ}i+(m+1)\right)\left(\left(-\text{Ᵽ}i+(m+1)\right)^2+a^2\right)}{((\text{Ᵽ}^2+(m+1)^2)^2-2a^2(\text{Ᵽ}^2-(m+1)^2))+a^4}$$

By the same method, we can prove that:

12) $Z^{c}\left((\ln(\varkappa))^{m}\sin\left(aln(\ln(\varkappa))\right)\right) = \int_1^e (\ln(\varkappa))^{m}\sin\left(aln(\ln(\varkappa))\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$

$$= \int_1^e \left(\frac{e^{aln(\ln(\varkappa)))i} - e^{-aln(\ln(\varkappa)))i}}{2i}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}(\ln(\varkappa))^{m}d\varkappa$$

$$= \int_1^e \left(\frac{e^{\ln\left((ln\varkappa)^{ai}\right)} - e^{\ln\left((ln\varkappa)^{-ai}\right)}}{2i}\right)(\ln\varkappa)^{m}\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$= \frac{1}{2i}\left(Z^{c}\left((ln\varkappa)^{m+ai}\right) - Z^{c}\left((ln\varkappa)^{m-ai}\right)\right)$$

$$= \frac{1}{2i}\left(\frac{1}{\text{Ᵽ}i+(m+ia+1)} + \frac{1}{\text{Ᵽ}i+(m-ia+1)}\right)$$

$$= \frac{1}{2i}\left(\frac{-2ia}{\left(\text{Ᵽ}i+(m+1)\right)^2+a^2}\right) = \frac{-a}{\left(\text{Ᵽ}i+(m+1)\right)^2+a^2}$$

$$= \frac{-a}{\left(\text{Ᵽ}i+(m+1)\right)^2+a^2}\times\frac{\left(-\text{Ᵽ}i+(m+1)\right)^2+a^2}{\left(-\text{Ᵽ}i+(m+1)\right)^2+a^2}$$

$$= \frac{-a\left(\left(-\text{Ᵽ}i+(m+1)\right)^2+a^2\right)}{((\text{Ᵽ}^2+(m+1)^2)^2-2a^2(\text{Ᵽ}^2-(m+1)^2)+a^4)}$$

13) $Z^{c}\left((\ln(\varkappa))^{m}\cosh\left(aln(\ln(\varkappa))\right)\right) = \int_1^e (\ln(\varkappa))^{m}\cosh\left(aln(\ln(\varkappa))\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$

$$= \int_1^e \left(\frac{e^{aln(\ln(\varkappa)))} + e^{-aln(\ln(\varkappa)))}}{2}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}(\ln(\varkappa))^{m}d\varkappa$$

$$= \int_1^e \left(\frac{e^{\ln((\ln(\varkappa))^{a})} + e^{\ln((\ln(\varkappa))^{-a})}}{2}\right)(\ln(\varkappa))^{m}\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$=\frac{1}{2}\left(Z^c((\ln(\varkappa))^{\text{ɱ}+a})+Z^c((\ln(\varkappa))^{\text{ɱ}-a})\right)$$

$$=\frac{1}{2}\left(\frac{1}{\text{Ᵽ}i+((\text{ɱ}+a)+1)}+\frac{1}{\text{Ᵽ}i+((\text{ɱ}-a)+1)}\right)$$

$$=\frac{1}{2}\left(\frac{2\text{Ᵽ}i+2\text{ɱ}+2}{-\text{Ᵽ}^2-a^2+2\text{ɱ}\text{Ᵽ}i+2\text{Ᵽ}i+\text{ɱ}^2+2\text{ɱ}+1}\right)$$

$$=\frac{(\text{Ᵽ}i+(\text{ɱ}+1))}{(i^2\text{Ᵽ}^2+2\text{Ᵽ}i(\text{ɱ}+1)+(\text{ɱ}+1)^2-a^2)}$$

$$=\frac{(\text{Ᵽ}i+(\text{ɱ}+1))}{\left((\text{Ᵽ}i+(\text{ɱ}+1))^2-a^2\right)}\times\frac{\left((-\text{Ᵽ}i+(\text{ɱ}+1))^2-a^2\right)}{\left((-\text{Ᵽ}i+(\text{ɱ}+1))^2-a^2\right)}$$

$$=\frac{(\text{ɱ}+1)\left((\text{Ᵽ}^2+(\text{ɱ}+1))^2-a^2\right)}{((\text{Ᵽ}^2+(\text{ɱ}+1)^2)^2+2a^2(\text{Ᵽ}^2-(\text{ɱ}+1)^2))+a^4}-\frac{\text{Ᵽ}i(\text{Ᵽ}^2+(1+\text{ɱ})^2+a^2)}{((\text{Ᵽ}^2+(1+\text{ɱ})^2)^2+2a^2(\text{Ᵽ}^2-(1+\text{ɱ})^2))+a^4}$$

14) $Z^c\left((\ln(\varkappa))^{\text{ɱ}}\sinh\left(aln(\ln(\varkappa))\right)\right)=\int_1^e(\ln(\varkappa))^{\text{ɱ}}\sinh\left(aln(\ln(\varkappa))\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$

$$=\int_1^e\left(\frac{e^{aln(\ln(\varkappa)))}-e^{-aln(\ln(\varkappa)))}}{2}\right)\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}(\ln(\varkappa))^m d\varkappa$$

$$=\int_1^e\left(\frac{e^{\ln((\ln(\varkappa))^a)}-e^{\ln((\ln(\varkappa))^{-a})}}{2}\right)(\ln(\varkappa))^{\text{ɱ}}\frac{(\ln(\varkappa))^{\text{Ᵽ}i}}{\varkappa}d\varkappa$$

$$=\frac{1}{2}\left(Z^c((\ln(\varkappa))^{\text{ɱ}+a})-Z^c((\ln(\varkappa))^{\text{ɱ}-a})\right)$$

$$=\frac{1}{2}\left(\frac{1}{\text{Ᵽ}i+(\text{ɱ}+a+1)}-\frac{1}{\text{Ᵽ}i+(\text{ɱ}-a+1)}\right)$$

$$=\frac{1}{2}\left(\frac{-2a}{(\text{Ᵽ}i+(\text{ɱ}+1))^2-a^2}\right)=\frac{-a}{(\text{Ᵽ}i+(\text{ɱ}+1))^2-a^2}$$

$$=\frac{-a}{(\text{Ᵽ}i+(\text{ɱ}+1))^2-a^2}\times\frac{(-\text{Ᵽ}i+(\text{ɱ}+1))^2-a^2}{(-\text{Ᵽ}i+(1+\text{ɱ}))^2-a^2}$$

$$=\frac{-a\left((-\text{Ᵽ}i+(\text{ɱ}+1))^2-a^2\right)}{((\text{Ᵽ}^2+(\text{ɱ}+1)^2)^2+2a^2(\text{Ᵽ}^2-(\text{ɱ}+1)^2)+a^4)}$$

$$=\frac{-a(-\text{Ᵽ}^2-2\text{Ᵽ}i(\text{ɱ}+1)+(\text{ɱ}+1)^2-a^2)}{((\text{Ᵽ}^2+(\text{ɱ}+1)^2)^2+2a^2(\text{Ᵽ}^2-(\text{ɱ}+1)^2)+a^4)}$$

$$=\frac{-a(-\text{Ᵽ}^2+(\text{ɱ}+1)^2-a^2)}{((\text{Ᵽ}^2+(\text{ɱ}+1)^2)^2+2a^2(\text{Ᵽ}^2-(\text{ɱ}+1)^2)+a^4)}+\frac{2a\text{Ᵽ}i(\text{ɱ}+1)}{((\text{Ᵽ}^2+(\text{ɱ}+1)^2)^2+2a^2(\text{Ᵽ}i^2-(\text{ɱ}+1)^2)+a^4)}$$

Examples (1.3):

1) $Z^c(-10)=\frac{-10}{1+p^2}+\frac{10p}{1+p^2}i$

2) $Z^c((\ln(\varkappa))^3)=\frac{4}{16+p^2}-\frac{ip}{16+p^2}$

3) $Z^c(-4\ln(\ln(\varkappa)))=4\frac{(1-ip)^2}{(1+p^2)^2}$

4) $Z^c\left(\sinh\left(5(\ln(\varkappa))\right)\right)=\frac{(120+5p^2)}{(1+p^2)^2-2(5)^2(1-p^2)+(5)^4}+\frac{2(5)pi}{(1+p^2)^2-2(5)^2(1-p^2)+(5)^4}$

$$=\frac{(120+5p^2)}{(1+p^2)^2-50(1-p^2)+625}+\frac{10pi}{(1+p^2)^2-50(1-p^2)+625}$$

5) $Z^c\left(\cosh\left(3(\ln(\varkappa))\right)\right)=\frac{1+p^2-(3)^2}{(3)^4+2(3)^2(p^2-1)+(p^2+1)^2}-\frac{(p+p^3+(3)^2p)i}{(3)^4+2(3)^2(p^2-1)+(p^2+1)^2}$

$$=\frac{p^2-8}{81+18(p^2-1)+(p^2+1)^2}-\frac{(p^3+10p)i}{81+18(p^2-1)+(p^2+1)^2}$$

6) $Z^c\left(\sin\left(-4(\ln(\varkappa))\right)\right)=\frac{64-4p^2+4}{256-32(p^2-1)+(p^2+1)^2}+\frac{-8pi}{256-32(p^2-1)+(p^2+1)^2}$

7) $Z^c\left(cos\left(2ln(\ln(\varkappa))\right)\right)=\frac{(p^2+5)}{16-8(p^2-1)+(p^2+1)^2}-\frac{\left((p^3-3p)\right)i}{16-8(p^2-1)+(p^2+1)^2}$

8) $Z^c(3(\ln(\varkappa))^2+4(\ln(\varkappa))^5+2(\ln(\varkappa))^{-3})=3Z^c((\ln(\varkappa))^2)+4Z^c((\ln(\varkappa))^5)+2Z^c((\ln(\varkappa))^{-3})$

$$=3\left(\frac{3}{9+p^2}-\frac{ip}{9+p^2}\right)+4\left(\frac{6}{36+p^2}-\frac{ip}{36+p^2}\right)+2\left(\frac{-2}{4+p^2}-\frac{ip}{4+p^2}\right)$$

$$=\left(\frac{9}{9+p^2}+\frac{24}{36+p^2}-\frac{4}{4+p^2}\right)+\left(-\frac{3ip}{9+p^2}-\frac{4ip}{36+p^2}-\frac{2ip}{4+p^2}\right)$$

9) $Z^c\left((\ln(\varkappa))^{-3}\sin\left(2ln(\ln(\varkappa))\right)\right)=\frac{-2\left((-ip+(-3+1))^2+4\right)}{((p^2+(-3+1)^2)^2-2(4)(p^2-(-3+1)^2)+16)}$

$$=\frac{\left(-2(-ip+(-3+1))^2-8\right)}{((p^2+4)^2-8(p^2-4)+16)}$$

10) $Z^c\left((\ln(\varkappa))^{\frac{3}{2}}\cos\left(3ln(\ln(\varkappa))\right)\right)=\frac{\left(ip+\left(\frac{5}{2}\right)\right)\left(\left(-ip+\left(\frac{5}{2}\right)\right)^2+9\right)}{\left(\left(p^2+\frac{25}{4}\right)^2-18\left(p^2-\frac{25}{4}\right)\right)+81}$

2. Conclusion

We conclude that it is possible to find transforms for some functions that operate in the field of complex numbers, capable of being used in other research by solving special types of differential equations, whether ordinary or partial.

Data Availability

The datasets used and analyzed during the current study are available from the corresponding author upon reasonable request.

Conflict of Interest

The authors declare no conflict of interest.

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