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Original Article

The Geometric Approach to Studying the Relations Between the Intervals of Uniqueness of Solutions Seventh -Order Differential Equation

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Abstract

This paper addresses the issue of the relations between semi-critical intervals of LHDE of the seventh order with (2, 3, 4, and 5 points) boundary conditions and with measurable coefficients. We shall use the geometric approach to state and prove some properties of LHDE. Moreover, the distribution of zeros in the solutions of the linear homogeneous differential equations (LHDE) has also been explored. The obtained results have been generalized for the sixth-order differential equation.

Keywords: Seventh order; Semi-oscillatory interval; Semi-critical interval; Boundary value problems; Fundamental normal solution; Linear differential equations; Distribution of zeros for the solution

1. Introduction

The studies of the distribution of the solution zeros of the LHDE began in the 60s of the last centuries. This field of study attracts many researchers and gains much more interest for its applications to functional equations [1], neutral differential equations [2, 3] differential equations with constant delay [4, 5], and variable delay [6]. The question of the laws of distribution of zeros of solutions of a linear differential equation touches upon many studies on the theory and practice of differential equations.

The following analogue of the Sturm theorem is obtained [7]: If the solutions u (x) and v (x) of the equation $y^{(n)}+$ g (x) y = 0 satisfy the conditions

$$u(a) = u'(a) = \cdots\cdots = u^{(n-2)}(a) = 0,\ u^{(n-1)}(a) = 1,\ \ u(b) = 0,$$

$$v(c) = v'(c) = \cdots\cdots = v^{(n-2)}(c) = 0,\ v^{(n-1)}(c) = 1,\ \ a < c < b,$$

then the solution v (x) does not have zeros in $(c, b]$.

Kondratiev [8] considered the same equation $y^{(n)} + g(x)y = 0$ for values of $n = 3$ and $n = 4$ at constant coefficient $g(x)$ proved alternation zeros of solutions.

Levin [9] showed that the theorem of. Kondratiev is also valid for equations of the form $y''' + g_1(x)y'' + g_2(x)y' = 0$, and $y^{(IV)} + (g(x)y')' = 0$.

After the publication of [10], where for a linear homogeneous differential equation of the third order of general form for $n = 3$ the laws of distribution of zeros of solutions were established in terms of semi-critical intervals, a large number of papers have appeared [7, 10-12]. They studied the problems of the distribution of zeros in the solutions of equations of (3,4,6) the order with summable coefficients besides the continuous ones. They used the geometric approach to prove the properties of the distribution of zeros in their solutions. The authors of [13-19] investigated LHDE of the (fifth, sixth, and seventh) order, they used the analytic approach to prove the properties of the distribution of LHDE.

In this paper, we shall use the geometric approach to state and prove some properties of LHDE of the seventh order with (2, 3, and 4 points) boundary conditions.

$$r_{61}(s) \geq r_{1111111}(s), r_{412}(s) \geq r_{1111111}(s), r_{3121}(s) \geq r_{1111111}(s).$$

2. Theoretical Framework

Consider the equation

$$L[y] = y^{(7)} - \sum_{j=0}^{6} g_j(x)y^{(j)} = 0, \qquad (1)$$

Assume that the coefficients $g_k(x)$ are measurable and continuous on $[a, b]$ satisfying the conditions

$$y^{(k_j)}(a_j) = A_{j,k_j},\ k_j = 0, \dots\dots, p_j - 1, j = 1, 2, \dots\dots, m,\ \sum_{j=0}^{m} p_j = 7, m \leq 7 \qquad (2)$$

where m is the number of points a_j, p_j is the number of conditions at the points a_j.

Problems (1) and (2) are called ≪ ($p_1 p_2 \dots p_m$ – problem) ≫.

Definition 2.1 [16]: For each fixed point $\propto\in[a,b)$, there exists a nonzero interval $[\propto,\beta)$, in which any non-trivial solution of equation (1) has no more than 6 zero, taking into account their multiplicities. This interval is called the semi-oscillation for equation (1). The maximum intervals of semi-oscillation with a common origin in $\propto$ is denoted by $[\propto, r(\propto))$.

Definition 2.2 [14]: The interval $[\propto, \mu)$, in which the given problem has a unique solution, which is called the semi-critical interval of this problem. The maximum intervals of semi-critical with a common origin at α are denoted by $[\propto, r_{p_1p_2\dots p_k}(\propto))$, k =2, 3, 4, 5,6.

The concept of the semi-critical interval is directly related to the distribution of zeros of the solution of equation (1).

We decipher the definitions of the maximal semi-critical intervals of some boundary value problems.

The interval $[\propto, r_{61}(\propto))$ is called such an interval in which any non-trivial solution (for the equation (1)) that has a zero at a_1 of multiplicity six and has no more zeros to the right of a_1, where $\propto\le a_1 < r_{61}(s) < a_2$.

In the interval $[\propto, r_{52}(\propto))$, nontrivial solution (for the equation (1)) that has a zero at a_1 of multiplicity five cannot have a double zero to the right of a_1, where:

$$\propto\le a_1 < r_{52}(\propto) < a_2.$$

A non-trivial solution (for the equation (1)) that has a zero at $a_1$of multiplicity three and has a double zero $a_2 > a_1$ cannot have a zero $a_3 > a_2$ of multiplicity higher than second in the interval $[\propto, r_{322}(\propto))$, where $\propto \le a_1 < a_2 < r_{322}(\propto) < a_3$.

In the interval $[\propto, r_{1111111}(\propto))$ a nontrivial solution cannot have seven different simple zeros.

The present paper considers the laws of the distribution of zeros. The main results of the distribution of zeros are as follows:

The interval $[s, r_{1111111}(s))$ is the intersection of intervals $[s, r_{p_1p_2\dots p_k}(s))$,

and is equal to the smallest of the intervals $r_{1111111}(s)\le \min\,[\,r_{p_1p_2\dots p_k}(s)]$

where

$$p_1+p_2+\dots+p_k=7,\ k=2, 3, 4, 5,6.$$

To prove the assertions formulated above, consider the following auxiliary lemmas.

Lemma 2.1 [15]: Let $v_1(x), v_2(x)$ – be a pair of not identically equal to zero, twice continuously differentiable functions, such that

$$v_1(x)\ne cv_2(x), (c = const), v_1(\alpha)=v_1(\beta)=0, v_2(x)\ne 0\ in\ [\propto,\ \beta].$$

Then there exists a linear combination

$$u(x)=c_1v_1(x)+c_2v_2(x), (c_1^2+c_2^2>0),$$

that

$$u(a)=U'(a)=0, \text{where } a\in(\propto,\ \beta).$$

Lemma 2.2 [15]: Let $v_1(x),\ v_2(x)$ – be a pair of not equal identically to zero (pair is not identically equal to zero), thrice continuously differentiable functions such that,

$$v_1(x)\ne cv_2(x), (c = const), v_1^{(k)}(a)=v_2^{(k)}(a)=0, k=0, 1.$$

Then, there exists a such linear combination

$$u(x)=c_1v_1(x)+c_2v_2(x),\ (c_1^2+c_2^2>0),$$

for which the point a is a three multiplicity zero, that is

$$u^{(i)}(a)=0, i=0,1,2.$$

Lemma 2.3 [15]: Let $v_1(x), v_2(x)$ – be twice continuously differentiable functions, satisfying the conditions $v_1(x).v_2(x)>0$, $\propto< x<\beta$,

$$v_1^{(k)}(a)=v_2^{(k)}(a),\ k=0,1; v_1''(a)\ne v_2''(a),\ a\in(\alpha,\beta);\ v_1(x)\ne v_2(x)\ \ (x\ =a);$$

then for any $\varepsilon>0$ there is a linear combination

$$u(x)=v_1(x)-cv_2(x),\ (c\ = const),$$

such that

$$u(a_1)=u(a_2)=0, \text{where } \propto< a_1<a<a_2<\beta, and\ max\ [|a-a_1|, |a-a_2|]<\varepsilon.$$

Lemma 2.4 [13]: Nontrivial solutions $v_1(x)$ and $v_2(x)$ of equation (1) are linearly dependent if

$$v_i^{(k)}(a)=0,\ k\ =0,1,2;\ i=1,2.$$

Lemma 2.5 [13]: Let $u(x)$, $v(x)$ – be a pair of non-trivial solutions of equation (1) such that

$$u^{(k)}(\alpha)=0, k=0,1,2,3,4,5,;\ v(\alpha)=0. \text{ If } u(x)\ne 0 \text{ in } (\propto,\ \beta+\varepsilon),$$

then for any $\varepsilon>0$ for some $c=const$ the difference $cu(x)-v(x)$ vanishes (goes to zero) at the points $b_i\in(\propto,\ \beta+\varepsilon)$, whose number is equal to $p+q$, p is the number of odd zeros of the solution $v(x)$ in $(\propto,\beta]$, , q is number of such $a_i\in[\alpha,\beta]$, that

$$v(a_i)=v'(\,a_i)=0,\ u'''(a)v''(a_i)>0.$$

3. Main Results

In this section, and according to the conditions which were mentioned in the above definitions, we will prove the following:

$$r_{1111111}(s)\le \min\,[\,r_{p_1p_2\dots p_k}(s)]$$

where

$$k=2, 3, 4, 5,6\ \ \ p_1+p_2+\dots+p_k=7.$$

Theorem 3.1: $r_{61}(s)\ge r_{1111111}(s)$.

Proof: Assume that $r_{61}(s)<r_{1111111}(s)$. Thus, two points exist

$$\propto,\beta\in[s, r_{1111111}(s)),$$

and a solution $u(x)$ of equation (1) in which the solution obeys

$$u(\propto)=u'(\propto)=u''(\propto)=u'''(\propto)=u^{(IV)}(\propto)=u^{(V)}(\propto)=u(\beta)=0, u(x)>0 \text{ in}$$

$$(\propto,\beta),$$

where

$$s\le\propto< r_{61}(s)<\beta<r_{1111111}(s)$$

and either

$$u'(\beta)\ne 0 \text{ or } u'(\beta)=u''(\beta)=0, \text{ either } u'(\beta)=0, u''(\beta)>0.$$

According to the above mentions, we consider two cases:

Case 1 β is a zero of odd multiplicity. We can assume that $\propto>s$.

Then, there exists a unique solution $v(x)$ of the equation (1) in the interval $[s,r_{1111111}(s))$ such that

$$v(a_i)=u(a_i), i=0,1,2,3,4,5,,6,\ v(a_7)>0,$$

where

$$s<a_0<\propto=a_1<a_2<a_3<a_4<a_5<a_6<r_{61}(s)<\beta=a_7<r_{1111111}(s),$$

Now by Lemma (2-3), the curves $u(x)$, $v(x)$ have no tangencies of even order in points with abscissas a_0,a_2,a_3,a_4,a_5,a_6. So, that either, $u'(a_0)<v'(a_0)$, or $u'(a_0)>v'(a_0)$, since by the condition $r_{61}(s)<\beta$, the curves $u(x)$ and $v(x)$ have not a fivefold tangency at the point with abscissa a_0 .

If

$$u'(a_0)<v'(a_0), v'(a_1)\ne 0, (a_i)=u(a_i), i=0,1,2,3,4,5,6 \text{ then } u'(a_5)>v'(a_5),$$

where

$$s < a_0 < \propto = a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < r_{61}(s) < \beta = a_7 < r_{1111111}(s),$$

$$v(a_7) > 0.$$

We note that curves $u(x)$, $v(x)$ have no tangencies of even order at points with abscissas $a_0, a_2, a_3, a_4, a_5, a_6$, but the difference $u(x) - v(x)$ has seven zeros $a_0, a_1, a_2, a_3, a_4, a_5, a_7$, in interval $[s, r_{1111111}(s))$ which is impossible.

If

$$u'(a_0) < v'(a_0), v'(a_0) \neq 0, , v(a_i) = u(a_i), i = 0,1,2,3,4,5,$$

then, $u'(a_5) > v'(a_5)$, hence, $u(x) > v(x)$ in some right half - neighborhood point a_5.

But, since $v(a_6)>0$, then the difference $u(x) - v(x)$ has zero at some point

$a' \in (a_5, a_6)$, which is impossible because this difference already has zeros at the five points $a_0, a_1, a_2, a_3, a_4, a_5$, in the interval $[s, r_{1111111}(s))$

where

$$s < a_0 < \propto = a_1 < a_2 < a_3 < a_4 < a_5 < a' < r_{61}(s) < \beta = a_6 < r_{1111111}(s).$$

If

$$u'(a_1) < v'(a_1),\ v(a_i) = u(a_i), i = 1,2,3,4,5,6,\ v(a_7) < 0$$

then curves $u(x), v(x)$ do not have a common point in the interval $[\beta, r_{1111111}(s))$, which means $u(x) - v(x) \neq 0$

in the interval $[\beta, r_{1111111}(s))$, now choose a point $a \in (\beta, r_{1111111}(s))$ such that

$u(a) < 0$, which yields that the linear combination

$$y(x) = u(x) - \frac{u(a)}{v(a)} v(x), \quad \text{for } c \to \frac{u(a)}{v(a)} \quad (\text{constant})$$

has a zero at the point a and at the six points $`a_1, `a_2, `a_3, `a_4, `a_5, `a_6 m$ in the interval $[s, r_{1111111}(s))$

which is impossible.

Finally, if

$$v'(\propto) = 0, v''(\propto) < 0,$$

then the difference $u(x) - v(x)$ has zeros at the points with abscissas a_0, a_2, a_3, a_4, a_5, and a double zero at the point $\propto = a_1$.

But, since the solution $w(x)$ has six - multiple zero at the point a_0, and has no zeros in the interval $(a_0, a_5 + \varepsilon), \varepsilon < r_{61} - a_5$, then, by Lemma (2.5) the linear combination

$cw(x) - [u(x) - v(x)]$ (for some values of *c= cost*)

has six zeros in the interval $(a_0, a_5 + \varepsilon) \subset [s, r_{1111111}(s))$, which is impossible.

So, if β - zero of odd multiplicity of the solution $u(x)$, which reveal that

$r_{61}(s) \geq r_{1111111}(s)$.

Case 2. β is a zero even multiplicity, there exist two points $\propto, \beta \in [s, r_{1111111}(s))$ and a solution $u(x)$ of equation (1) such that

$u(\propto) = u'(\propto) = u''(\propto) = u'''(\propto) = u^{(4)}(\propto) = u(\beta) = u'(\beta) = 0, u(x) > 0$, in $(\propto, \beta)$,

where

$$s \leq \propto < r_{61}(s) < \beta < r_{1111111}(s), u''(\propto) > 0,\ u''(\beta) > 0.$$

Consider the solution $v(x)$ satisfactorily boundary conditions

$$u(a_i) = v(a_i),\ i = 0,1,2,3,4,5, v'(\propto) > 0, v(\beta) > 0,$$

where

$$s < a_0 < \propto = a_1 < a_2 < a_3 < a_4 < a_5 < r_{61}(s) < \beta = a_6 < r_{1111111}(s),$$

and the curves $u(x), v(x)$ do not have a common point in the interval $[\beta, r_{1111111}(s))$.

It means $u(x) - v(x) \neq 0$ in the interval $[\beta, r_{1111111}(s))$, then, choose the point

$a \in (\beta, r_{1111111}(s))$, where $v(a) > 0$, the difference

$$y(x) = u(x) - \frac{u(a)}{v(a)} v(x)$$

has zero at the point a and six zeros in points $a_0, `a_1, `a_2, `a_3, `a_4, `a_5$, where $`a_1, `a_2, `a_3, `a_4, `a_5$, are the shifted positions of the points a_1, a_2, a_3, a_4, a_5, for $c \to \frac{u(a)}{v(a)}$, which is impossible.

Consider the solution of $v(x)$ such that

$$u(a_i) = v(a_i),\ i = 0,1,2,3,4,5, v(a_6) > 0,$$

where

$$s < a_0 < \propto = a_1 < a_2 < a_3 < a_4 < a_5 < r_{61}(s) < \beta = a_6 < r_{1111111}(s).$$

As mentioned above via Lemma (2.5) the curves $u(x)$ and $v(x)$ have no tangencies at the abscissa points a_1, a_2, a_3, a_4, a_5.

There $v'(\propto) < 0$ or $v'(\propto) = 0$, and $v''(\propto) < 0$,

then the contradiction with the assumption is obvious, for then $u'(a_5) > v'(a_5)$.

Hence, the difference $u(x) - v(x)$ has zero at some point $b \in (a_5, a_6)$, since by construction $v(a_6)> 0$, which is impossible because this difference already has six zero at the points $a_0, a_1, a_2, a_3, a_4, a_5 \in [s, r_{1111111}(s))$.

If $u(a_i) = v(a_i),\ i = 0,1,2,3,4,5,\ v(a_6) > 0,\ v(\propto) = v'(\propto) = 0,$ $v''(\propto) > 0$,

then the difference $u(x) - v(x)$ has zeros at the abscissas a_0, a_2, a_3, a_4, a_5 points and a double zero at the point $\propto = a_1$.

But, since the solution $w(x)$ has three - multiple zero at the point a_0, and no zeros in region $(a_0, a_5 + \varepsilon)$, $\varepsilon < r_{61} - a_5$.

Thereby, depending on Lemma (2.5), there would be a linear combination

$cw(x) - [u(x) - v(x)]$ for some value of $c = const$

associated with seven zeros in the region$(a_0, a_5 + \varepsilon) \subset [s, r_{1111111}(s))$, which is impossible, leading to $r_{61}(s) \geq r_{1111111}(s)$. The theorem is proved.

Theorem 3.2: $r_{412}(s) \geq r_{1111111}(s)$

Proof: Assume that $r_{412}(s) < r_{1111111}(s)$, which means the existence of a solution $u(x)$ of the equation (1) having three zeros $\propto, \beta, \gamma$ in the interval $[s, r_{1111111}(s))$ such that

$$u(\propto) = u'(\propto) = u''(\propto) = u'''(\propto) = u(\beta) = u(\gamma) = u'(\gamma),$$

where

$$u'''(\propto)u''(\gamma) \neq 0.$$

Since $u(x) \neq 0$ in the interval $]\propto, \beta[\cup]\beta, \gamma[$. Since $u(x) < 0$ in the interval $(\gamma, r_{1111111}(s))$.

Consider the solution $v(x)$ satisfying the boundary conditions

$$u(a_i) = v(a_i),\ i=1,2,3,4,5,6, v(\gamma) < 0,$$

where

$$s < \propto = a_1 < a_2 < a_3 < \beta < a_4 < a_5 < a_6 < r_{312}(s) < \gamma < r_{1111111}(s),$$

by virtue of Lemma (2.3) and Theorem (3.1), the curves $u(x), v(x)$ have no tangencies at the points $a_1, a_2, a_3, a_4, a_5, a_6$, with abscissas in the interval $[s, r_{1111111}(s))$.

Therefore, if $v'(\propto) > 0$ and $v(\gamma) < 0$, then putting $c = \frac{u(a)}{v(a)}$

where $\gamma < a < r_{1111111}(s)$ it is easy to make sure that the difference $u(x) - cv(x)$

has seven zeros $`a_1$, $`a_2$, $`a_3$, $`a_4$, $`a_5$, $`a_5$, a, in the interval $[s, r_{1111111}(s))$ where $`a_1$, $`a_2$, $`a_3$, $`a_4$, $`a_5$, $`a_5$ are the shifted positions of the points $a_1, a_2, a_3, a_4, a_5, a_6$,

for $c = \frac{u(a)}{v(a)}$, which is impossible.

fore, if

The impossibility of the inequality $v'(\propto) > 0$ is almost obvious.

If

$$u(a_i) = v(a_i),\ i=0,1,2,3,4,5,6,\ v'(\propto) = 0,\ v''(\propto) < u''(\propto),$$

$$s < \propto = a_1 < a_2 < a_3 < \beta < a_4 < a_5 < a_6 < r_{312}(s) < \gamma < r_{1111111}(s),$$

Then $v'(a_6) > u'(a_6)$, consequently, by virtue of the condition $v\ (\gamma) < 0$, the difference

$u(x) - v(x)$ has a zero $a \in (a_6, \gamma)$, which is impossible, because the solution of the form

$$y(x) = u(x) - v(x)$$

have seven zeros $a_1, a_2, a_3, a_4, a_5, a_6, a$, in the interval $[s, r_{1111111}(s))$.

Similarly, the contradiction exits in the case of following condition:

Finally, if

$$v'(\propto) = 0 \text{ and } v''(\propto) > u''(\propto),$$

then, by virtue of Lemma (2.2) a linear combination

$$u(x) - cv(x) \text{ (at some c = cost)}$$

having three - multiple zero at the point a_0 and zeros $`a_1$, $`a_2$, $`a_3$ $`a_4$, are the shifted positions of the points a_1, a_2, a_3, a_4, for $c \to \frac{u''(a_0)}{v''(a_0)}$, which contradicts Theorem (3.1). The theorem is proved.

Theorem 3.3: $r_{3121}(s) \geq r_{111111}(s)$

Proof: Assume that $r_{3121}(s) < r_{111111}(s)$,

then, there is a pair of non - trivial solutions $u(x)$, $v(x)$ of equation (1)

such that

$$u(a_1) = u'(a_1) = u''(a_1) = u(a_2) = u(a_3) = u'(a_3) = u(a_4) = 0,$$

and

$$u^{(k)}(a_1) = 0,\ k = 0,1,2,3,4,5,\quad \text{sgn}\, v'''(a_1) = \text{sgn}\, u''(a_3),\ v(a_4) > 0,$$

where

$$s < a_1 < a_2 < a_3 < r_{3121}(s) < a_4 < r_{1111111}(s).$$

By virtue of Lemmas (2.5) and Theorem (3.1),

$$u''(a_3) \neq 0,\ sgnu''(a_1) = -sgnu''(a_3), u'(a_2)u'(a_4) < 0$$

which mean that

$$u'(a_2) = -sgnu'(a_4), v(x) \neq 0 \text{ in the interval } (a_1, a_4+\varepsilon).$$

It is easy making sure that the difference

$$u(x) - \frac{u(a)}{v(a)}v(x),\ (a_3 < a < a_4),$$

has seven zeroes in the interval $[s, r_{1111111}(s))$, which is impossible.

This contradiction proves the theorem, $r_{3121}(s) \geq r_{1111111}(s)$.

Which is a prove of the suggested theory.

Theorem 3.4: $r_{31111}(s) \geq r_{1111111}(s)$.

Proof: Obtain the validity of the assertion of the Theorem, using Lemma (2.1) and

Theorem (3.3).

Indeed, if assume that

$$r_{31111}(s) < r_{1111111}(s),$$

then some solution $u(x)$ of the equation (1) has in the interval $[s, r_{1111111}(s))$

six consecutive zeroes $a_1, a_2, a_3, a_4, a_5, a_6$, the first of which double zero, such that

$$u(a_1) = u'(a_1) = u''(a_1) = u(a_2) = u(a_3) = u(a_4) = u(a_5) = 0,$$

note that $u'(a_5) > 0$, mean $\text{sgn}\, u'(a_2) = -\text{sgn}\, u'(a_5)$.

If $v(x)$ is the solution of equation (1) where

$$v^{(k)}(a_1) = 0,\ k = 0,1,2, u(b_i) = v(b_i), i = 1,2,3,$$

where

$$s < a_1 = b_1 < b_2 < a_2 < a_3 < b_3 < a_4 < r_{31111}(s) < a_5 < r_{1111111}(s),$$

$$\text{sgn}\, v'''(a_1) = sgn\, u''(a_1), v'(b_2) < u'(b_2)\ and\ v'(b_3) > u'(b_3),$$

then, by virtue of Lemma (2.1) there exists a linear combination

$$z(x) = c_1u(x) + c_2v(x),\ (c_1^2 + c_2^2 > 0)$$

has zeros at the points with abscissas $`b_2$, $`b_3$, where $`b_2$, $`b_3$ are the shifted positions of the points b_2, b_3, and a double zero at the point c, ($where\ c \in (a_2, a_3)$) and also has a zero of multiplicity three at the point a_1, This means

$$z(a_1) = z'(a_1) = z''(a_1) = z(`b_2) = z(c) = z'(c) = z(`b_3) = 0.$$

Then, we have arrived to a contradiction with Theorem (3.3) ≪(3121 – problem)≫

This contradiction proves the theorem of $r_{31111}(s) \geq r_{1111111}(s)$.

The theorem is proved

In the same way, we could prove the remaining formulas, thereby, we extracted the following results

$$r_{1111111}(s) \leq \min\,[\, r_{p_1 p_2 \dots p_k}(s)]$$

where

$$p_1 + p_2 + \dots + p_k = 7, k = 2,3,4,5,6$$

4. Conclusions

This study investigates the distribution of zeros of non-trivial solutions of a linear homogeneous equation of seventh order in terms of semi-critical intervals of boundary value problems and the description of the behavior trend of the estimated intervals of uniqueness solutions. Basically, we have obtained new results (Theorems 3.1, 3.2, 3.3, 3.4) using these theorems for establishing the limiting relations between the lengths of semi-critical intervals of the uniqueness of solutions (two, three, four, and five points) boundary value problems with fixed points, and the description of their estimated behavior. In this work, we prove that the interval $[s, r_{1111111}(s))$ is the intersection of the following intervals $[s, r_{61}(s))$, $[s, r_{412}(s))$, $[s, r_{3121}(s))$, $[s, r_{31111}(s))$.

Data Availability

The datasets used and analyzed during the current study are available from the corresponding author upon reasonable request.

Competing Interests

The author has declared that no competing interests exist.

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