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Some Coefficient Estimates for Subclass of Starlike Functions Associated with the Sine Function Defined by Subordination

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Abstract:

In this article, we investigate the logarithmic coefficients for a subclass of starlike functions with respect to symmetric conjugate points associated with the sine function. Although this class has been previously studied in the context of coefficient bounds and geometric properties, the logarithmic coefficients, especially higher-order ones, have not been extensively addressed in the literature. We derive explicit formulas for the first six logarithmic coefficients γ_1 through γ_6 for functions in this class. We establish precise bounds for the Hankel coefficients, Hankel determinants $H_{2,1}(f), H_{2,2}(f), H_{3,1}(f)$ and $H_{4,1}(f)$ associated with the class $\mathcal{S}_{SC}^{*}(\sin z)$. In addition, we derive sharp estimates for the Hankel determinant for the Logarithmic coefficients $H_{2,1}(F_f/2)$ and $H_{2,2}(F_f/2)$ within the same class.

Keywords: Analytic functions; Starlike functions; Coefficient estimates; Logarithmic coefficients; Subordination; Hankel determinant

1. Introduction and preliminaries

This article lies within the framework of geometric function theory, with a particular focus on the study of starlike functions- an important class of univalent analytic functions defined in the open unit disk.

The study of logarithmic coefficients, especially those of higher order, plays a vital role in understanding the growth, distortion, and geometric behavior of analytic functions.

In this work, we consider a subclass of starlike functions with respect to symmetric conjugate points associated with the sine function. The sine function, being a classical transcendental function, possesses rich analytic and geometric properties that motivate the present investigation.

Let $\mathcal{A}$ denote the class of functions f that are analytic in the unit disk

$D = \{z : z \in \mathbb{C}\ and\ |z| < 1\}$, and that has a Maclaurin series expansion of the form

$$\mathrm{f(z)} = \mathrm{z} + \sum_{m=2}^{\infty} a_m z^m \quad , \quad (\mathrm{z} \in D). \tag{1.1}$$

A subclass of $\mathcal{A}$, denoted by $\mathcal{S}$, consists of functions that are univalent and normalized such that $f(0) = 0, f'(0) = 1$. Let $\mathcal{S}^*$denote the subclass of $\mathcal{S}$ consisting of starlike functions, i.e., $f \in \mathcal{S}^*$ if and only if:

$$\mathcal{R}e\,\frac{zf'(z)}{f(z)} > 0 \ \ , z \in D. \tag{1.2}$$

Let $\mathcal{B}$ denote the family of Schwarz functions $w(z)$, that are analytic in D given by

$$w(z) = \sum_{n=1}^{\infty} b_n z^n \quad , \quad (\mathrm{z} \in D)\,,$$

and satisfying $w(0) = 0$ and $|w(z)|$ <1 for all $z \in D$. Given analytic functions f and g in D, we say that f is subordinated to g, written $f \prec g$, if there exists a Schwarz function $w(z)$ such that $f(z) = g\big(w(z)\big)\,, z \in D$.

When g is univalent and $f(0) = g(0)$, then $f(D) \subset g(D)$.

For fixed constants A and B satisfying $-1 \le B < A \le 1$, denoted by $P\,[A, B]$, the family of functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n\,.$$

A function $p(z)$, analytic in the unit disk D, belongs to the Janowski class $P\,[A, B]$ if and only if

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad (\mathrm{z} \in D),$$

where $w(z)$ is the Schwarz function. This class $P\,[A, B]$ is known as the Janowski class and was introduced by Janowski [7].

Class $P\,[A, B] \subset P\,[1, -1] = P$, then it reduces to the class P, the well-known class of functions with positive real part consists of functions p that satisfy $\mathcal{R}e\ p(z) > 0$ and $p\ (0) = 1$.

We now introduce a subclass of starlike functions with respect to symmetric conjugate points connected to the sine function as follows:

Definition 1.1.

Let $\mathcal{S}_{SC}^{*}(\sin z)$ denote the class of analytic functions f satisfying the subordination condition:

$$\frac{zf'(z)}{h(z)} \prec \varphi(z), z \in D, \tag{1.3}$$

where $h(z)=\frac{f(z)-\overline{f(-\bar{z})}}{2}$ and $\varphi(z)=1+\sin z$.

It is worth noting that several studies, including those by El-Ashwah and Thomas [6] and Ping and Janteng [18] have investigated subclasses of starlike functions with respect to symmetric conjugate points, particularly using classical subordination conditions such as $\frac{1+z}{1-z}$ or Janowski-type functions, i.e.,

$$\mathcal{S}_{SC}^{*}=\left\{f\in\mathcal{A}:\mathcal{R}e\left(\frac{zf'(z)}{h(z)}\right)>0\,,z\in D\right\},$$

and

$$\mathcal{S}_{SC}^{*}[A,B]=\left\{f\in\mathcal{A}:\mathcal{R}e\left(\frac{zf'(z)}{h(z)}\right)\prec\frac{1+Az}{1+Bz}\,,-1\le B<A\le 1\,,z\in D\right\}.$$

However, most of these works have focused on coefficient estimates, geometric properties, or second Hankel determinants. In contrast, this work introduces a distinct subclass $\mathcal{S}_{SC}^{*}(\sin z)$, defined by subordination associated with the sine function $\varphi(z)=1+\sin z$, and presents a comprehensive study of the logarithmic coefficients γ_1 to γ_6 as well as higher-order Hankel determinants $H_{2,1}(F_f/2)$, $H_{2,2}(F_f/2)$, $H_{2,1}(f), H_{2,2}(f), H_{3,1}(f)$ and $H_{4,1}(f)$ while the general framework aligns with prior literature in geometric function theory, this article distinguishes itself by providing new sharp bounds for both Taylor coefficients up to and logarithmic coefficients, which have received limited attention in previous research. This extension bridges a gap in the literature and contributes to a deeper understanding of analytic behavior in sine-associated subclasses $\mathcal{S}_{SC}^{*}$ and $\mathcal{S}_{SC}^{*}[A,B]$.

In 2023, Mohamad et al. [15] introduced the subclass of star-like functions with respect to symmetric conjugate points associated with the sine function. Some coefficient functionals for this class are considered. Bounds of Taylor coefficients, logarithmic coefficients, and the Hankel and Toeplitz determinants whose entries are logarithmic coefficients are provided. Comparison with Mohamad et al. (2023):

1- Main Similarity:

Both articles study a subclass of starlike analytic functions with respect to symmetric conjugate points associated with the sine function. They use a similar subordination condition of the form: $\frac{zf'(z)}{h(z)}\prec\varphi(z), z\in D$.

2- Key Differences: It can be tabulated in the following table (Table 1).

Table 1: Key differences of comparison with Mohamad et al. (2023).

Aspect	Current work	Mohamad *et al.* (2023) [15]
Order of coefficients	Computes Taylor coefficient up to a_7, logarithmic coefficients up to γ_6, and Hankel determinants up to the fourth order for Taylor and second logarithmic coefficients	Mainly deals with lower order, typically up to a_5, logarithmic coefficients up to γ_4, and Hankel determinants up to the second logarithmic coefficients.
Depth of Analysis	Offers a more comprehensive and detailed investigation of higher-order coefficients, including Hankel determinants.	Focuses on a specific class with limited analysis of the coefficient order.
Scientific Contribution	Presents new and original results that extend beyond previous works by including higher-order terms and advanced determinant Analysis	Provides a complementary study within a narrower, symmetry-based framework.

The Fekete-Szegö inequality is a well-known result concerning the coefficients of univalent analytic functions, initially formulated by Fekete and Szegö in 1933 in connection with the Bieberbach conjecture. A related and important problem in the theory of univalent functions is the study of Hankel determinants, which have proven helpful in the investigation of singularities and power series with integral coefficients.

For the functions $f\in\mathcal{A}$ of the form (1.1), in 1976, Noonan and Thomas [16] stated the ℓ^{th} Hankel determinant as

$$H_{\ell,n}(f)=\begin{vmatrix}a_n & a_{n+1} & \cdots & a_{n+\ell-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+\ell-2}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n+\ell-1} & a_{n+\ell} & \cdots & a_{n+2(\ell-1)}\end{vmatrix}, \tag{1.4}$$

$(a_1 = 1\ \ell, n\in N=\{1,2,\cdots.\})$.

In particular, we have

$$H_{2,1}(f)=\begin{vmatrix}a_1 & a_2\\ a_2 & a_3\end{vmatrix}=a_3-a_2{}^2\ (a_1=1, n=1, \ell=2)\,, \tag{1.5}$$

$$H_{2,2}(f)=\begin{vmatrix}a_2 & a_3\\ a_3 & a_4\end{vmatrix}=a_2a_4-a_3{}^2\ (\ n=2, \ell=2), \tag{1.6}$$

$$H_{3,1}(f)=\begin{vmatrix}a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\end{vmatrix}=a_3a_5-a_3^3-a_4^2-a_2^2a_5+2a_2a_3a_4$$
$$=a_3H_{2,2}(f)+a_4I+a_5H_{2,1}(f), \tag{1.7}$$

where $I=a_2a_3-a_4$, and

$$H_{4,1}(f)=\begin{vmatrix}a_1 & a_2 & a_3 & a_4\\ a_2 & a_3 & a_4 & a_5\\ a_3 & a_4 & a_5 & a_6\\ a_4 & a_5 & a_6 & a_7\end{vmatrix}$$
$$=-a_4\begin{vmatrix}a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\\ a_4 & a_5 & a_6\end{vmatrix}+a_5\begin{vmatrix}a_1 & a_3 & a_4\\ a_2 & a_4 & a_5\\ a_3 & a_5 & a_6\end{vmatrix}$$
$$-a_6\begin{vmatrix}a_1 & a_2 & a_4\\ a_2 & a_3 & a_5\\ a_3 & a_4 & a_6\end{vmatrix}+a_7\begin{vmatrix}a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\end{vmatrix}$$
$$=-a_4[a_4(a_3a_5-a_4^2)-a_5(a_2a_5-a_3a_4)+a_6(a_2a_4-a_3{}^2\ )]$$
$$+a_5[a_3(a_3a_5-a_4^2)-a_5(a_5-a_2a_4)+a_6(a_4-a_2a_3)]$$
$$-a_6[a_3(a_2a_5-a_3a_4)-a_4(a_5-a_2a_4)+a_6(a_3-a_2{}^2\ )]+a_7H_{3,1}(f)$$
$$=a_7H_{3,1}(f)-a_6\rho_1+a_5\rho_2-a_4\rho_3, \tag{1.8}$$

where $\rho_1=a_3(a_2a_5-a_3a_4)-a_4(a_5-a_2a_4)+a_6(a_3-a_2{}^2\ )$,

$$\rho_2=a_3(a_3a_5-a_4^2)-a_5(a_5-a_2a_4)+a_6(a_4-a_2a_3),$$
$$\rho_3=a_4(a_3a_5-a_4^2)-a_5(a_2a_5-a_3a_4)+a_6(a_2a_4-a_3{}^2\ ).$$

We note that $H_{2,1}(f)$ is the well-known Fekete-Szegö functional [10], which is generalized as

$$\nabla(\mu,f\ )=|a_3\ -\ \mu a_2{}^2\ |, \tag{1.9}$$

for $\mu\ \in\ \mathbb{C}$.

Recently, the Hankel determinants of a function $f\in\mathcal{A}$ whose elements are logarithmic coefficients of $f\in\mathcal{A}$ have been introduced by Kowalczyk and Lecko [10, 11]

$$H_{\ell,n}(F_f/2)=\begin{vmatrix}\gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+\ell-1}\\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+\ell-2}\\ \vdots & \vdots & \cdots & \vdots\\ \gamma_{n+\ell-1} & \gamma_{n+\ell} & \cdots & \gamma_{n+2(\ell-1)}\end{vmatrix}.$$

The logarithmic coefficients are defined in the series form

$$\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}\gamma_nz^n\ . \tag{1.10}$$

Taking both sides and using (1.1) or differentiating (1.10) and using (1.1), we get

$$\gamma_1=\frac{1}{2}a_2\ \ , \tag{1.11}$$

$$\gamma_2=\frac{1}{2}\Big(a_3-\frac{1}{2}a_2^2\Big), \tag{1.12}$$

$$\gamma_3=\frac{1}{2}\Big(a_4-a_2a_3+\frac{1}{3}a_2^3\Big), \tag{1.13}$$

$$\gamma_4=\frac{1}{2}\Big(a_5-a_2a_4+a_2^2a_3-\frac{1}{2}a_3^2-\frac{1}{4}a_2^4\Big), \tag{1.14}$$

$$\gamma_5=\frac{1}{2}\Big(a_6-a_2a_5-a_3a_4+a_2^2a_4+a_2a_3^2-a_3a_2^3+\frac{1}{5}a_2^5\Big), \tag{1.15}$$

$$\gamma_6=\frac{1}{2}\Big(a_7-a_2a_6-a_3a_5+a_2^2a_5-\frac{3}{2}a_2^2a_3^2$$
$$-a_4a_2^3-\frac{1}{2}a_4^2+2a_2a_3a_4+\frac{1}{3}\,a_3^3+a_3a_2^4$$
$$-\frac{1}{6}a_2^6\Big). \tag{1.16}$$

The logarithmic coefficients have great importance; for instance, these coefficients helped Kayumov [8] to solve Brennan's conjecture for conformal mapping, and the estimation of the logarithmic coefficients can be transferred to the Taylor coefficients of univalent functions via the Lebedev–Milin inequalities [4] (for details).

Some recent works on this problem that relate to the theory of univalent functions have been studied in [1, 14, 18], but only a few articles have been published for the class of starlike functions with respect to other points. Motivated by these works, in this article, we obtain the upper bounds of the Taylor coefficients $|a_n|, n=2,3,4,5,6,7$.

In recent years, many articles have been devoted to finding the upper bounds for the second-order Hankel determinant $H_{2,2}$, for various subclasses of analytic functions and the upper bounds for the third and fourth-order Hankel determinants by many researchers [9,12,17,19,20]. Recently, Cho et al. [3] introduced the following function class $\mathcal{S}_s^*$

$$\mathcal{S}_s^* = \left\{ f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \prec 1 + \sin z, (z \in \mathrm{D}) \right\},$$

which implies that the quantity $\frac{zf'(z)}{f(z)}$ lies in an eight-shaped region in the right-half plane. The work investigates a subclass of starlike functions with respect to symmetric conjugate points associated with the sine function. The sine function is a classical transcendental function with rich analytic and geometric properties.

The study extends theoretical knowledge by deriving sharp estimates and properties of higher-order logarithmic coefficients, and by connecting classical functions to complex analysis by associating the sine function with a subclass of starlike functions.

The article opens up new directions for analyzing function classes that are both geometrically meaningful and analytically rich.

Although the study is primarily theoretical, it has potential indirect applications in areas such as:

- Control theory and analytic transforms
- Signal and image analysis
- Complex differential equations
- Approximation theory and numerical analysis

2. Preliminary results

In this section, we give some lemmas to prove our main results.

Lemma 2.1. ([4]) For a function $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D$

the sharp inequality $|c_n| \le 2$ holds for each $n \ge 1$ and

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|^2}{2}.$$

Equality holds for the function $p(z) = \frac{1+z}{1-z}$.

Lemma 2.2. ([5]) Let $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D$ and $\mu \in \mathbb{C}$. Then

$$|c_n - \mu c_k c_{n-k}| \le 2 \max\{1, |2\mu - 1|\}, 1 \le k \le n-1.$$

If $|2\mu - 1| \ge 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations.

If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$ or its rotations.

Lemma 2.3. ([2]) Let $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D$ and $\alpha, \beta, \delta \in \mathbb{R}$. Then

$$|\alpha c_1^3 - \beta c_1 + \gamma c_3| \leqslant 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \delta|.$$

Lemma 2.4. [13] If $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D$, then

$$|c_2 - \mu c_1^2| \le \begin{cases} -4\mu + 2 & if\ \mu \le 0 \\ 2 & if\ 0 \le \mu \le 1 \\ 4\mu - 2 & \text{if}\ \mu \ge 1 \end{cases}$$

When $\mu < 0$ or $\mu > 1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \mu < 1$, then equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\mu = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \le \lambda \le 1)$$

Or one of its rotations. If $\mu = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that the equality holds in the case of $\mu = 0$.

3. Taylor coefficients and Fekete-Szegö inequality for $f \in \mathcal{S}_{SC}^*(\sin z)$

Theorem 3.1. If f is of the form (1.1) belongs to $\mathcal{S}_{SC}^*(\sin z)$, then

$$|a_2| \le \frac{1}{2}, |a_3| \le \frac{1}{2},$$

$$|a_4| \le \frac{1}{4}, |a_5| \le \frac{1}{2}$$

$$|a_6| \le \frac{1009}{1440}, |a_7| \le \frac{31}{9},$$

and

$$|a_3 - \mu a_2^2| \le \frac{1}{4}\begin{cases} -4v + 2 & if\ v \le 0 \\ 2 & if\ 0 \le v \le 1 \\ 4v - 2 & \text{if}\ v \ge 1 \end{cases},$$

where $v = \frac{1}{2}\left(1 + \frac{\mu}{2}\right)$.

Proof. Since $f \in \mathcal{S}_{SC}^*(\sin z)$, from the definition of subordination, there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$, and from (1.3) we have

$$\frac{zf'(z)}{h(z)} = 1 + \sin w(z), z \in D. \tag{3.1}$$

Assuming that

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

$$1 + w(z) = p(z)\big(1 - w(z)\big) \Rightarrow w(z)\big(1 + p(z)\big) = p(z) - 1.$$

This leads to

$$\begin{aligned} w(z) = \frac{p(z) - 1}{p(z) + 1} &= \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots} \\ &= \frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}{c_1}^2\right)z^2 \\ &+ \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{1}{4}{c_1}^3\right)z^3 \\ &+ \frac{1}{2}\left(c_4 - c_1 c_3 - \frac{1}{2}{c_2}^2 + \frac{3}{4}{c_1}^2 c_2 - \frac{1}{8}{c_1}^4\right)z^4 \\ &+ \frac{1}{2}\left(c_5 - c_1 c_4 - c_2 c_3 - \frac{1}{2}c_2 {c_1}^3 \right. \\ &\left. + \frac{3}{4}{c_2}^2 c_1 + \frac{3}{4}c_3 {c_1}^2 + \frac{1}{16}{c_1}^5\right)z^5 \\ &+ \frac{1}{2}\left(c_6 - c_1 c_5 - c_2 c_4 - \frac{1}{2}{c_3}^2 \right. \\ &+ \frac{3}{2}c_1 c_2 c_3 + \frac{3}{4}c_4 {c_1}^2 + \frac{1}{4}{c_2}^3 - \frac{3}{4}{c_1}^2 {c_2}^2 \\ &\left. + \frac{5}{16}{c_1}^4 c_2 - \frac{1}{32}{c_1}^6\right)z^6 + \cdots. \end{aligned}$$

Hence, from the right-hand side of (3.1), we obtain

$$\begin{aligned} 1 + \sin w(z) = 1 + w(z) - \frac{\big(w(z)\big)^3}{3!} + \frac{\big(w(z)\big)^5}{5!} - \cdots \\ &= 1 + \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}{c_1}^2\right)z^2 \\ &+ \left(\frac{5{c_1}^3}{48} - \frac{c_1 c_2}{2} + \frac{c_3}{2}\right)z^3 \\ &+ \left(\frac{c_4}{2} + \frac{5{c_1}^2 c_2}{16} - \frac{{c_2}^2}{4} - \frac{c_1 c_3}{2} - \frac{{c_1}^4}{32}\right)z^4 \\ &+ \left(\frac{c_5 - c_1 c_4 - c_2 c_3}{2} - \frac{1}{8}c_2 {c_1}^3 \right. \\ &\left. + \frac{5{c_2}^2 c_1 + 5 c_3 {c_1}^2}{16} + \frac{1}{3840}{c_1}^5\right)z^5 \\ &+ \left(\frac{c_6 - c_1 c_5 - c_2 c_4}{2} - \frac{1}{4}{c_3}^2 + \frac{5}{8}c_1 c_2 c_3 + \frac{5}{48}{c_2}^3 \right. \\ &+ \frac{1}{32}{c_1}^4 c_2 - \frac{1}{8}c_2 {c_1}^3 + \frac{5}{16}{c_1}^2 c_4 - \frac{3}{16}{c_1}^2 {c_2}^2 \\ &\left. + \frac{1}{96}{c_1}^6\right)z^6 + \cdots. \end{aligned}$$

On the other hand, since f of the form (1.1), this gives

$$zf'(z) = z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + 5a_5 z^5 + \cdots,$$

and

$$h(z) = \frac{2z + \sum_{n=2}^{\infty}(1 - (-1)^n)a_n z^n}{2} = z + a_3 z^3 + a_5 z^5 + a_7 z^7 + \cdots.$$

Further, we have from (3.1) that

$$zf'(z) = h(z)(1 + \sin w(z)),$$

$$z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + 5a_5 z^5 + 6a_6 z^6 + 7a_7 z^7 + \cdots$$

$$
\begin{aligned}
&= (z + a_3 z^3 + a_5 z^5 + a_7 z^7 + \cdots) \\
&\quad \cdot \Big[1 + \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right) z^2 \\
&\quad + \left(\frac{5c_1^3}{48} - \frac{c_1 c_2}{2} + \frac{c_3}{2}\right) z^3 \\
&\quad + \frac{1}{2}\left(c_4 - c_1 c_3 - \frac{1}{2}c_2^2 + \frac{5}{8}c_1^2 c_2 - \frac{1}{16}c_1^4\right) z^4 \\
&\quad + \Big(\frac{c_5 - c_1 c_4 - c_2 c_3}{2} - \frac{1}{8}c_2 c_1^3 \\
&\quad + \frac{5c_2^2 c_1 + 5c_3 c_1^2}{16} + \frac{1}{3840}c_1^5\Big) z^5 \\
&\quad + \Big(\frac{c_6 - c_1 c_5 - c_2 c_4}{2} - \frac{1}{4}c_3^2 + \frac{5}{8}c_1 c_2 c_3 + \frac{5}{48}c_2^3 \\
&\quad + \frac{1}{32}c_1^4 c_2 - \frac{1}{8}c_2 c_1^3 + \frac{5}{16}c_1^2 c_4 - \frac{3}{16}c_1^2 c_2^2 \\
&\quad + \frac{1}{96}c_1^6\Big) z^6 + \cdots\Big]. \qquad (3.2)
\end{aligned}
$$

Expanding the series and comparing the coefficients of z^n, n = 1, 2, 3, 4, 5,6,7 on both sides of (3 2) yields

$$2a_2 = \frac{1}{2}c_1 \Rightarrow a_2 = \frac{c_1}{4}, \qquad (3.3)$$

$$\frac{c_2}{2} - \frac{c_1^2}{4} + a_3 = 3a_3$$

$$\Rightarrow a_3 = \frac{1}{8}(2c_2 - c_1^2), \qquad (3.4)$$

$$\frac{5c_1^3}{48} - \frac{c_1 c_2}{2} + \frac{c_3}{2} + \frac{c_1 a_3}{2} = 4a_4 \Rightarrow a_4 = \frac{c_1^3}{96} - \frac{3c_1 c_2}{32} + \frac{c_3}{8}$$

$$\Rightarrow a_4 = \frac{1}{96}(c_1^3 - 9c_1 c_2 + 12c_3), \qquad (3.5)$$

$$a_5 + \frac{c_4}{2} + \frac{5c_1^2 c_2}{16} - \frac{c_2^2}{4} - \frac{c_1 c_3}{2} - \frac{c_1^4}{32} + a_3\left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) = 5a_5$$

$$\Rightarrow a_5 = \frac{c_4}{8} + \frac{3c_1^2 c_2}{64} - \frac{c_2^2}{32} - \frac{c_1 c_3}{8}$$

$$\Rightarrow a_5 = \frac{1}{64}(8c_4 + 3c_1^2 c_2 - 2c_2^2 - 8c_1 c_3), \qquad (3.6)$$

$$
\begin{aligned}
6a_6 &= \frac{c_5 - c_1 c_4 - c_2 c_3}{2} - \frac{1}{8}c_2 c_1^3 + \frac{5c_2^2 c_1 + 5c_3 c_1^2}{16} + \frac{1}{3840}c_1^5 \\
&\quad + a_3\left(\frac{5c_1^3}{48} - \frac{c_1 c_2}{2} + \frac{c_3}{2}\right) + \frac{1}{2}a_5 c_1, \qquad (3.7)
\end{aligned}
$$

Substituting (3.4) and (3.6) in (3.7), then we get

$$6a_6 = \frac{1}{2}c_5 - \frac{7}{16}c_1 c_4 - \frac{3}{8}c_2 c_3 + \frac{3}{16}c_3 c_1^2 - \frac{5}{384}c_2 c_1^3 + \frac{11}{64}c_1^2 c_2 - \frac{49}{3840}c_1^5$$

$$
\begin{aligned}
\Rightarrow a_6 &= \frac{1}{12}c_5 - \frac{7}{96}c_1 c_4 - \frac{1}{16}c_2 c_3 + \frac{3}{96}c_3 c_1^2 - \frac{5}{2304}c_2 c_1^3 + \frac{11}{384}c_1^2 c_2 \\
&\quad - \frac{49}{23040}c_1^5
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow a_6 \\
&= \frac{1}{23040}(1920c_5 - 1680c_1 c_4 - 1440c_2 c_3 + 720c_3 c_1^2 - 50c_2 c_1^3 \\
&\quad + 660c_2^2 c_1 - 49c_1^5) \\
&= \frac{1}{23040}\Big[1920\left(c_5 - \frac{7}{8}c_1 c_4\right) - 50c_2\left(c_1^3 - \frac{66}{5}c_1 c_2 + \frac{144}{5}c_3\right) \\
&\quad - 720c_1^2\left(\frac{49}{720}c_1^3 - c_3\right)\Big], \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
7a_7 &= \frac{c_6 - c_1 c_5 - c_2 c_4}{2} - \frac{1}{4}c_3^2 + \frac{5}{8}c_1 c_2 c_3 + \frac{5}{48}c_2^3 \\
&\quad + \frac{1}{32}c_1^4 c_2 - \frac{1}{8}c_3 c_1^3 + \frac{5}{16}c_1^2 c_4 - \frac{3}{16}c_1^2 c_2^2 \\
&\quad + \frac{1}{96}c_1^6 + a_3\left(\frac{c_4}{2} + \frac{5c_1^2 c_2}{16} - \frac{c_2^2}{4} - \frac{c_1 c_3}{2} - \frac{c_1^4}{32}\right) \\
&\quad + a_5\left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right) + a_7, \qquad (3.9)
\end{aligned}
$$

Substituting (3.4) and (3.6) in (3.9), then we get

$$
\begin{aligned}
6a_7 &= \frac{c_6 - c_1 c_5}{2} - \frac{5}{16}c_2 c_4 - \frac{1}{4}c_3^2 + \frac{7}{16}c_1 c_2 c_3 + \frac{5}{192}c_2^3 \\
&\quad + \frac{113}{256}c_1^4 c_2 - \frac{1}{32}c_3 c_1^3 + \frac{7}{32}c_1^2 c_4 - \frac{3}{64}c_1^2 c_2^2 \\
&\quad + \frac{11}{768}c_1^6
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow a_7 &= \frac{1}{6}\Big(\frac{c_6 - c_1 c_5}{2} - \frac{5}{16}c_2 c_4 - \frac{1}{4}c_3^2 + \frac{7}{16}c_1 c_2 c_3 + \frac{5}{192}c_2^3 \\
&\quad + \frac{113}{256}c_1^4 c_2 - \frac{1}{32}c_3 c_1^3 + \frac{7}{32}c_1^2 c_4 - \frac{3}{64}c_1^2 c_2^2 \\
&\quad + \frac{11}{768}c_1^6\Big)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow a_7 &= \frac{1}{6}\Big[\frac{1}{2}(c_6 - c_1 c_5) - \frac{1}{32}c_3(c_1^3 - 14c_1 c_2 + 8c_3) \\
&\quad + \frac{7}{32}c_1^2\left(c_4 - \frac{3}{14}c_2\right)^2 + \frac{113}{256}c_1^4\left(c_2 + \frac{11}{339}c_1^2\right) \\
&\quad - \frac{5}{16}c_2\left(c_4 - \frac{1}{12}c_2^2\right)\Big]. \qquad (3.10)
\end{aligned}
$$

Using triangle inequality and Lemma 2.1 in (3.3), we get

$$|a_2| \le \frac{1}{2}.$$

Now, applying Lemma 2.4 in (3.4) and Lemma 2.3 in (3.5), respectively, implies that

$$|a_3| = \frac{1}{8}|2c_2 - c_1^2| = \frac{1}{4}\left|c_2 - \frac{c_1^2}{2}\right| \le \frac{1}{2},$$

where $\mu = \frac{1}{2}$.

$$
\begin{aligned}
|a_4| &= \frac{1}{96}|c_1^3 - 9c_1 c_2 + 12c_3| \\
&\le \frac{1}{96}[2|1| + 2|9 - 2(1)| + 2|1 - 9 + 12|] \le \frac{1}{4},
\end{aligned}
$$

where $\alpha = 1$, $\beta = 9$, $\delta = 12$.

Rearranging the terms in (3.6), we can rewrite it as

$$a_5 = \frac{1}{64}\left(8(c_4 - c_1 c_3) - 2c_2\left(c_2 - \frac{3}{2}c_1^2\right)\right),$$

$$|a_5| = \frac{1}{64}|8(c_4 - w_1 c_1 c_3) - 2c_2(c_2 - w_2 c_1^2)|,$$

where $w_1 = 1$ and $w_2 = \frac{3}{2}$.

Consequently, by applying Lemma 2.1, Lemma 2.2, and Lemma 2.4 as well as the triangle inequality, we obtain

$$|a_5| \le \frac{1}{64}\left[8 \cdot 2\max\{1,1\} + 16\right] = \frac{1}{2}.$$

Using triangle inequality and Lemmas 2.1, 2.2, and 2.3 in (3. 8), we get

$$
\begin{aligned}
|a_6| &\le \frac{1}{23040}\Big[1920 \cdot 2\max\left\{1, \left|2 \cdot \frac{7}{8} - 1\right|\right\} + 50 \\
&\quad \cdot 2\left(2|1| + 2\left|\frac{66}{5} - 2\right| + 2\left|1 - \frac{66}{5} + \frac{144}{5}\right|\right) + 720 \\
&\quad \cdot 4\left(2\left|\frac{49}{720}\right| + 2\left|(-2) \cdot \frac{49}{720}\right| + 2\left|\frac{49}{720} - 1\right|\right)\Big] \\
&= \frac{1}{23040}\Big[3840 + 100\left(2 + \frac{112}{5} + \frac{166}{5}\right) \\
&\quad + 2880\left(\frac{49}{360} + \frac{49}{180} + \frac{671}{360}\right)\Big]
\end{aligned}
$$

$$= \frac{1}{23040}(3840 + 5760 + 6544) = \frac{16144}{23040} = \frac{1009}{1440},$$

where $\mu = \frac{7}{8}$, $\alpha_1 = 1$, $\beta_1 = \frac{66}{5}$, $\delta_1 = \frac{144}{5}$, $\alpha_2 = \frac{49}{720}$, $\beta_2 = 0$, $\delta_2 = -1$.

Using triangle inequality and Lemma 2.1, Lemma 2.2, Lemma 2.3, and Lemma 2.4 in (3. 10), we get

$$
\begin{aligned}
|a_7| &\le \frac{1}{6}\Big[\frac{1}{2}(2) + \frac{1}{32} \cdot 2(2 + 2|14 - 2| + 2|1 - 14 + 8|) + \frac{7}{32} \cdot 4 \\
&\quad \cdot 2\max\left\{1, \left|2 \cdot \frac{3}{14} - 1\right|\right\} + \frac{113}{256} \cdot 16 \\
&\quad \cdot 2\max\left\{1, \left|2\left(-\frac{11}{339}\right) - 1\right|\right\} + \frac{5}{16} \\
&\quad \cdot 2\max\left\{1, \left|2 \cdot \frac{1}{12} - 1\right|\right\}\Big]
\end{aligned}
$$

$$= \frac{1}{6}\left(1 + \frac{9}{4} + \frac{7}{4} + \frac{5}{8} + \frac{361}{24}\right) = \frac{496}{144} = \frac{31}{9},$$

where $\mu_1 = 1$, $\alpha_1 = 1$, $\beta_1 = 14$, $\delta_1 = 8$, $\mu_2 = \frac{3}{14}$, $\mu_3 = -\frac{11}{339}$, $\mu_4 = \frac{1}{12}$.

$$a_3 - \mu a_2^2 = \frac{1}{8}(2c_2 - c_1^2) - \mu\frac{c_1^2}{16} = \frac{1}{4}\left\{c_2 - \frac{1}{2}\left(1 + \frac{\mu}{2}\right)c_1^2\right\},$$

where $v = \frac{1}{2}\left(1 + \frac{\mu}{2}\right)$.

Using Lemma 2.4, we get

$$|a_3 - \mu a_2^2| \le \frac{1}{4}\begin{cases} -4v + 2 & \text{if } v \le 0 \\ 2 & \text{if } 0 \le v \le 1 \\ 4v - 2 & \text{if } v \ge 1 \end{cases}.$$

This completes the proof of Theorem 3.1.

4. Hankel determinant for the Logarithmic coefficients of $f \in \mathcal{S}^*_{SC}(\sin z)$

Theorem 4.1. If f is of the form (1.1) belongs to $\mathcal{S}^*_{SC}(\sin z)$, then

$$|\gamma_1| \le \frac{1}{4}\ , |\gamma_2| \le \frac{1}{4}\ , |\gamma_3| \le \frac{1}{8}\ ,$$

$$|\gamma_4| \le \frac{7}{16}\ , |\gamma_5| \le \frac{9937}{5760}\ , |\gamma_6| \le \frac{23}{9}.$$

Proof. Putting (3.3) - (3.6) in (1.11) - (1.14), we obtain

$$\gamma_1 = \frac{a_2}{2} = \frac{c_1}{8}, \tag{4.1}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}{a_2}^2\right) = \frac{1}{2}\left[\frac{1}{8}(2c_2 - {c_1}^2) - \frac{1}{2}\left(\frac{c_1}{4}\right)^2\right] = \frac{1}{2}\left(\frac{c_2}{4} - \frac{5{c_1}^2}{32}\right)$$
$$= \frac{1}{8}\left(c_2 - \frac{5}{8}{c_1}^2\right), \tag{4.2}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}{a_2}^3\right)$$
$$= \frac{1}{2}\left[\frac{1}{96}({c_1}^3 - 9c_2 + 12c_3) - \left(\frac{c_1}{4}\right)\left(\frac{1}{8}(2c_2 - {c_1}^2)\right) + \frac{1}{3}\left(\frac{c_1}{4}\right)^3\right]$$
$$= \frac{1}{2}\left(\frac{{c_1}^3}{96} - \frac{3c_1c_2}{32} + \frac{c_3}{8} - \frac{c_1c_2}{16} + \frac{{c_1}^3}{32} + \frac{{c_1}^3}{192}\right)$$
$$= \frac{1}{2}\left(\frac{3{c_1}^3}{64} - \frac{5c_1c_2}{32} + \frac{c_3}{8}\right) = \frac{3{c_1}^3}{128} - \frac{5c_1c_2}{64} + \frac{c_3}{16}$$
$$= \frac{1}{128}(3{c_1}^3 - 10c_1c_2 + 8c_3), \tag{4.3}$$

$$\gamma_4 = \frac{1}{2}\left[\frac{1}{64}(8c_4 + 3{c_1}^2c_2 - 2{c_2}^2 - 8c_1c_3) - \frac{c_1}{4}\left(\frac{1}{96}({c_1}^3 - 9c_1c_2 + 12c_3)\right) + \left(\frac{c_1}{4}\right)^2\left(\frac{1}{8}(2c_2 - {c_1}^2)\right) - \frac{1}{2}\left(\frac{1}{8}(2c_2 - {c_1}^2)\right)^2 - \frac{1}{4}\left(\frac{c_1}{4}\right)^4\right]$$
$$= \frac{1}{2}\left(\frac{11{c_1}^2c_2}{128} + \frac{c_4}{8} - \frac{{c_2}^2}{32} - \frac{5c_1c_3}{32} - \frac{4{c_1}^4}{384} - \frac{{c_1}^4}{1024} - \frac{{c_2}^2}{32} + \frac{{c_1}^2c_2}{32} - \frac{{c_1}^4}{128}\right)$$
$$= \frac{c_4}{16} + \frac{15{c_1}^2c_2}{256} - \frac{{c_2}^2}{32} - \frac{5c_1c_3}{64} - \frac{59{c_1}^4}{6144}$$
$$= \frac{1}{6144}(384c_4 - 192{c_2}^2 - 480c_1c_3 + 360{c_1}^2c_2 - 59{c_1}^4), \tag{4.4}$$

$$\gamma_5 = \frac{1}{2}\left(a_6 - a_2a_5 - a_3a_4 + a_2^2a_4 + a_2a_3^2 - a_3a_2^3 + \frac{1}{5}a_2^5\right)$$
$$= \frac{1}{2}\Big[\frac{1}{23040}(1920c_5 - 1680c_1c_4 - 1440c_2c_3 + 720c_3{c_1}^2 - 50c_2{c_1}^3 + 660{c_2}^2c_1 - 49{c_1}^5)$$
$$- \frac{1}{256}(8c_1c_4 + 3{c_1}^3c_2 - 2c_1{c_2}^2 - 8{c_1}^2c_3)$$
$$- \frac{1}{768}(2{c_1}^3c_2 - 18{c_2}^2c_1 + 24c_2c_3 - {c_1}^5 + 9{c_1}^3c_2 - 24c_3{c_1}^2) + \frac{1}{1536}({c_1}^5 - 9{c_1}^3c_2 + 12c_3{c_1}^2)$$
$$+ \frac{1}{256}(4{c_2}^2c_1 + {c_1}^5 - 4{c_1}^3c_2) - \frac{1}{512}(2{c_1}^3c_2 - {c_1}^5) + \frac{1}{5120}{c_1}^5\Big]$$
$$= \frac{1}{2}\left(\frac{1}{12}c_5 - \frac{13}{32}c_1c_4 - \frac{19}{96}c_2c_3 + \frac{47}{640}c_3{c_1}^2 - \frac{33}{512}c_2{c_1}^3 + \frac{41}{640}{c_2}^2c_1 - \frac{271}{46080}{c_1}^5\right)$$
$$= \frac{1}{2}\left[\frac{1}{12}\left(c_5 - \frac{39}{8}c_1c_4\right) - \frac{33}{512}c_2\left({c_1}^3 - \frac{164}{165}c_1c_2 + \frac{304}{99}c_3\right) - \frac{271}{46080}{c_1}^2\left({c_1}^3 - \frac{3384}{271}c_3\right)\right], \tag{4.5}$$

$$\gamma_6 = \frac{1}{2}\left(a_7 - a_2a_6 - a_3a_5 + a_2^2a_5 - \frac{3}{2}a_2^2a_3^2 - a_4a_2^3 - \frac{1}{2}a_4^2 + 2a_2a_3a_4 + \frac{1}{3}a_3^3 + a_3a_2^4 - \frac{1}{6}a_2^6\right)$$
$$= \frac{1}{2}\Big[\frac{1}{6}\Big(\frac{c_6 - c_1c_5}{2} - \frac{5}{16}c_2c_4 - \frac{1}{4}{c_3}^2 + \frac{7}{16}c_1c_2c_3 + \frac{5}{192}{c_2}^3 + \frac{113}{256}{c_1}^4c_2 - \frac{1}{32}c_3{c_1}^3 + \frac{7}{32}{c_1}^2c_4 - \frac{3}{64}{c_1}^2{c_2}^2 + \frac{11}{768}{c_1}^6\Big)$$
$$- \frac{1}{92160}(1920c_1c_5 - 1680{c_1}^2c_4 - 1440c_1c_2c_3 + 720c_3{c_1}^3 - 50c_2{c_1}^4 + 660{c_2}^2{c_1}^2 - 49{c_1}^6)$$
$$- \frac{1}{512}(16c_2c_4 + 8{c_1}^2{c_2}^2 - 4{c_2}^3 - 16c_1c_2c_3 - 8{c_1}^2c_4 - 3{c_1}^4c_2 + 8c_3{c_1}^3)$$
$$+ \frac{1}{1024}\left(8c_4{c_1}^2 + 3{c_1}^4c_2 - 2{c_1}^2{c_2}^2 - 8{c_1}^3c_3\right)$$
$$- \frac{3}{2048}(4{c_1}^2{c_2}^2 + {c_1}^6 - 4{c_1}^4c_2)$$
$$- \frac{1}{6144}({c_1}^6 - 9{c_1}^4c_2 + 12c_3{c_1}^3)$$
$$- \frac{1}{18432}\left({c_1}^6 + 144{c_3}^2 + 81{c_1}^2{c_2}^2 - 216c_1c_2c_3 + 24c_3{c_1}^3 - 18{c_1}^4c_2\right)$$
$$+ \frac{1}{1536}\left(11{c_1}^4c_2 + 24c_1c_2c_3 - 18{c_1}^2{c_2}^2 - {c_1}^6 - 12c_3{c_1}^3\right)$$
$$+ \frac{1}{1536}\left(8{c_2}^3 - 12{c_1}^2{c_2}^2 + 6{c_1}^4c_2 - {c_1}^6\right)$$
$$+ \frac{1}{2048}(2{c_1}^4c_2 - {c_1}^6) - \frac{1}{24576}{c_1}^6\Big]$$
$$= \frac{1}{2}\Big(\frac{1}{12}c_6 - \frac{5}{48}c_1c_5 - \frac{1}{12}c_2c_4 + \frac{113}{768}c_1c_2c_3 + \frac{5}{288}{c_2}^3 - \frac{19}{384}{c_3}^2 + \frac{1903}{18432}{c_1}^4c_2 - \frac{73}{1536}c_3{c_1}^3 + \frac{5}{64}{c_1}^2c_4 - \frac{383}{6144}{c_1}^2{c_2}^2 + \frac{107}{122880}{c_1}^6\Big)$$
$$= \frac{1}{2}\Big[\frac{1}{12}\left(c_6 - \frac{5}{4}c_1c_5\right) - \frac{73}{1536}c_3\left({c_1}^3 - \frac{226}{73}c_1c_2 + \frac{76}{73}c_3\right) + \frac{5}{64}{c_1}^2\left(c_4 - \frac{383}{480}{c_2}^2\right) + \frac{1903}{18432}{c_1}^4\left(c_2 + \frac{321}{38060}{c_1}^2\right) - \frac{1}{12}c_2\left(c_4 - \frac{5}{24}{c_2}^2\right)\Big]. \tag{4.6}$$

The bounds of $|\gamma_1|$, $|\gamma_2|$, $|\gamma_3|$, $|\gamma_4|$, $|\gamma_5|$ *and* $|\gamma_6|$ follow from Lemma 2.1, Lemma 2.2, Lemma 2.3, and Lemma 2.4. On the other hand, rearranging the terms in (4.1), (4.2), (4.3), (4.4), (4.5), and (4.6) we get

$$|\gamma_1| = \left|\frac{c_1}{8}\right| \le \frac{2}{8} = \frac{1}{4},$$

$$|\gamma_2| = \frac{1}{8}\left|c_2 - \frac{5}{8}{c_1}^2\right| \le \frac{1}{8}\cdot 2 = \frac{1}{4},$$

where $\mu = \frac{5}{8}$.

$$|\gamma_3| = \frac{1}{128}|3{c_1}^3 - 10c_1c_2 + 8c_3|$$
$$\le \frac{1}{128}[2|3| + 2|10 - 6| + 2|3 - 10 + 8|] = \frac{16}{128} = \frac{1}{8},$$

where $\alpha = 3, \beta = 10, \delta = 8$.

$$|\gamma_4| = \frac{1}{6144}|384c_4 - 192{c_2}^2 - 480c_1c_3 + 360{c_1}^2c_2 - 59{c_1}^4|$$
$$= \frac{1}{6144}\left|384\left(c_4 - \frac{1}{2}{c_2}^2\right) - c_1(480c_3 - 360c_1c_2 + 59{c_1}^3)\right|$$
$$\le \frac{1}{6144}\Big[384\cdot 2\max\left\{1, \left|2\left(\frac{1}{2}\right) - 1\right|\right\} + 2(2|59| + 2|360 - 2(59)| + 2|59 - 360 + 480|)\Big] = \frac{1}{6144}(768 + 1920)$$
$$= \frac{2688}{6144} = \frac{7}{16},$$

where $\mu = \frac{1}{2}, \alpha = 59, \beta = 360, \delta = 480$.

$$|\gamma_5| = \frac{1}{2}\left|\left[\frac{1}{12}\left(c_5 - \frac{39}{8}c_1c_4\right) - \frac{33}{512}c_2\left({c_1}^3 - \frac{164}{165}c_1c_2 + \frac{304}{99}c_3\right) - \frac{271}{46080}{c_1}^2\left({c_1}^3 - \frac{3384}{271}c_3\right)\right]\right|$$

$$\le \frac{1}{2}\Big[\frac{1}{12}\cdot 2\max\Big\{1,\Big|2\Big(\frac{39}{8}\Big)-1\Big|\Big\}+2\cdot\frac{33}{512}\Big(2|1|+2\Big|\frac{164}{165}-2(1)\Big|+2\Big|1-\frac{164}{165}+\frac{304}{99}\Big|\Big)+\frac{73}{15360}\cdot 4\Big(2|1|+2|0-2(1)|+2\Big|1-0-\frac{3384}{271}\Big|\Big)\Big]$$

$$=\frac{1}{2}\Big[\frac{1}{6}\cdot\frac{35}{4}+\frac{33}{256}\Big(2+\frac{332}{165}+\frac{3046}{495}\Big)+\frac{271}{11520}\Big(2+4+\frac{6226}{271}\Big)\Big]$$

$$=\frac{1}{2}\Big(\frac{35}{24}+\frac{33}{256}\cdot\frac{5032}{495}+\frac{271}{11520}\cdot\frac{7852}{271}\Big)=\frac{1}{2}\Big(\frac{35}{24}+\frac{629}{480}+\frac{1963}{2880}\Big)=\frac{1}{2}\cdot\frac{9937}{2880}=\frac{9937}{5760},$$

where $\mu=\frac{39}{8}$, $\alpha_1=1, \beta_1=\frac{164}{165}$, $\delta_1=\frac{304}{99}$, $\alpha_2=1, \beta_2=0$, $\delta_2=-\frac{3384}{271}$.

$$|\gamma_6|=\frac{1}{2}\Big|\Big[\frac{1}{12}\Big(c_6-\frac{5}{4}c_1c_5\Big)-\frac{73}{1536}c_3\Big(c_1^3-\frac{226}{73}c_1c_2+\frac{76}{73}c_3\Big)+\frac{5}{64}c_1^2\Big(c_4-\frac{383}{480}c_2^2\Big)+\frac{1903}{18432}c_1^4\Big(c_2+\frac{321}{38060}c_1^2\Big)-\frac{1}{12}c_2\Big(c_4-\frac{5}{24}c_2^2\Big)\Big]\Big|$$

$$\le\frac{1}{2}\Big[\frac{1}{12}\cdot 2\max\Big\{1,\Big|2\Big(\frac{5}{4}\Big)-1\Big|\Big\}+2\cdot\frac{73}{1536}\Big(2|1|+2\Big|\frac{226}{73}-2(1)\Big|+2\Big|1-\frac{226}{73}+\frac{76}{73}\Big|\Big)+\frac{5}{64}\cdot 4\cdot 2\max\Big\{1,\Big|2\Big(\frac{383}{480}\Big)-1\Big|\Big\}+\frac{1903}{18432}\cdot 16\cdot 2\max\Big\{1,\Big|2\Big(\frac{-321}{38060}\Big)-1\Big|\Big\}+\frac{1}{12}\cdot 2\cdot 2\max\Big\{1,\Big|2\Big(\frac{5}{24}\Big)-1\Big|\Big\}\Big]$$

$$=\frac{1}{2}\Big[\frac{1}{4}+\frac{73}{768}\Big(2+\frac{160}{73}+\frac{154}{73}\Big)+\frac{5}{8}+\frac{1903}{576}+\frac{1}{3}\Big]=\frac{1}{2}\Big[\frac{1}{4}+\frac{460}{768}+\frac{5}{8}+\frac{1903}{576}+\frac{1}{3}\Big]=\frac{1}{2}\cdot\frac{46}{9}=\frac{23}{9},$$

where $\mu_1=\frac{5}{4}$, $\alpha_1=1, \beta_1=\frac{226}{73}$, $\delta_1=\frac{76}{73}$, $\mu_2=\frac{383}{480}$, $\mu_3=\frac{-321}{38060}$, $\mu_4=\frac{5}{24}$.

Theorem 4.2. If f is of the form (1.1) belongs to $\mathcal{S}_{SC}^*(\sin z)$, then

$$|H_{2,1}(F_f/2)|\le\frac{75}{512}.$$

Proof: In view of (4.1), (4.2), (4.3), we have

$$H_{2,1}(F_f/2)=\gamma_1\gamma_3-\gamma_2^2=\frac{c_1}{8}\Big(\frac{1}{128}(3c_1^3-10c_1c_2+8c_3)\Big)-\Big(\frac{1}{8}\Big(c_2-\frac{5}{8}c_1^2\Big)\Big)^2$$

$$=\frac{1}{64}\Big(\frac{3}{16}c_1^4-\frac{10}{16}c_1^2c_2+\frac{1}{2}c_1c_3-c_2^2+\frac{5}{4}c_1^2c_2-\frac{25}{64}c_1^4\Big)$$

$$=\frac{1}{4096}(-13c_1^4+40c_1^2c_2+32c_1c_3-64c_2^2). \tag{4.7}$$

Rearranging the terms in (4.7), it becomes

$$\gamma_1\gamma_3-\gamma_2^2=\frac{1}{4096}(-c_1(13c_1^3-40c_1c_2-32c_3)-64c_2^2),$$

where $\chi=13, \lambda=40$, and $\eta=-32$.

By applying the triangle inequality as well as Lemma 2.1 and Lemma 2.3, we get the desired inequality.

$$|\gamma_1\gamma_3-\gamma_2^2|\le\frac{1}{4096}([2(2|13|+2|40-26|+2|13-40-32|)]+64(4))$$

$$=\frac{1}{4096}(2(172)+64(4))=\frac{600}{4096}\le\frac{75}{512}.$$

Theorem 4.3. If f is of the form (1.1) belongs to $\mathcal{S}_{SC}^*(\sin z)$, then

$$|H_{2,2}(F_f/2)|\le\frac{33}{256}.$$

Proof: In view of (4.2), (4.3), (4.4), we have

$$H_{2,2}(F_f/2)=\gamma_2\gamma_4-\gamma_3^2$$

$$=\Big(\frac{1}{8}\Big(c_2-\frac{5}{8}c_1^2\Big)\Big)\Big(\frac{1}{6144}(384c_4-192c_2^2-480c_1c_3+360c_1^2c_2-59c_1^4)\Big)-\Big(\frac{1}{128}(3c_1^3-10c_1c_2+8c_3)\Big)^2$$

$$=\frac{1}{49152}\Big[\Big(384c_4c_2-192c_2^3-480c_1c_2c_3+480c_1^2c_2^2-284c_1^4c_2-240c_1^2c_4+300c_1^3c_3+\frac{295}{8}c_1^6\Big)-\frac{1}{16384}(9c_1^6-60c_1^4c_2+100c_1^2c_2^2+64c_3^2+48c_1^3c_3-160c_1c_2c_3)\Big]$$

$$=\frac{1}{128}c_4c_2-\frac{1}{256}c_2^3+\frac{15}{4096}c_1^2c_2^2-\frac{13}{6144}c_1^4c_2-\frac{5}{1024}c_1^2c_4+\frac{13}{4096}c_1^3c_3+\frac{79}{393216}c_1^6-\frac{1}{256}c_3^2$$

$$=\Big[\frac{1}{128}c_4\Big(c_2-\frac{5}{8}c_1^2\Big)-\frac{1}{256}c_2^2\Big(c_2-\frac{15}{16}c_1^2\Big)+\frac{79}{393216}c_1^3\Big(c_1^3-\frac{832}{79}c_1c_2+\frac{1248}{79}c_3\Big)-\frac{1}{256}c_3^2\Big],$$

$$|H_{2,2}(F_f/2)|\le\frac{1}{128}\cdot 4+\frac{1}{256}\cdot 8+\frac{79}{393216}\cdot 8\Big(2|1|+2\Big|\frac{832}{79}-2(1)\Big|+2\Big|1-\frac{832}{79}+\frac{1248}{79}\Big|\Big)+\frac{1}{256}\cdot 4$$

$$=\frac{1}{16}+\frac{13}{256}+\frac{1}{64}=\frac{33}{256}.$$

5. Hankel determinant for the Taylor coefficients of $f\in\mathcal{S}_{SC}^*(\sin z)$

Corollary 5.1. If f is of the form (1.1) belongs to $\mathcal{S}_{SC}^*(\sin z)$, then

$$|H_{2,1}(f)|\le\frac{1}{2}$$

Proof:

Putting $\mu=1$ In Theorem 3.1, we obtain

$$|H_{2,1}(f)|=|a_3-a_2^2|\le\frac{1}{4}\cdot 2=\frac{1}{2},$$

where $v=\frac{1}{2}\Big(1+\frac{1}{2}\Big)=\frac{3}{4}$, $0<\frac{3}{4}<1$.

Theorem 5.2. If f is of the form (1.1) belongs to $\mathcal{S}_{SC}^*(\sin z)$, then

$$|H_{3,1}(f)|\le\frac{29}{48}.$$

Proof: In view of (3.3), (3.4), (3.5), we have

$$H_{2,2}(f)=a_2a_4-a_3^2=-\frac{5}{384}c_1^4+\frac{1}{32}c_1c_3+\frac{1}{128}c_1^2c_2-\frac{1}{16}c_2^2$$

$$=-\frac{5}{384}c_1\Big(c_1^3-\frac{3}{5}c_1c_2-\frac{12}{5}c_3\Big)-\frac{1}{16}c_2^2.$$

Using the triangle inequality and Lemmas 2.1 and 2.3, we obtain

$$|H_{2,2}(f)|\le\frac{5}{384}\cdot 2\Big(2|1|+2\Big|\frac{3}{5}-2(1)\Big|+2\Big|1-\frac{3}{5}-\frac{12}{5}\Big|\Big)+\frac{1}{16}\cdot 4$$

$$=\frac{11}{48}+\frac{1}{4}=\frac{23}{48}. \tag{5.1}$$

$$|a_4-a_2a_3|=\frac{1}{24}\Big|c_1^3-\frac{3}{4}c_1c_2+3c_3\Big|\le\frac{1}{24}\Big(2|1|+2\Big|\frac{3}{4}-2(1)\Big|+2\Big|1-\frac{3}{4}+3\Big|\Big)=\frac{11}{24} \tag{5.2}$$

Using the triangle inequality, Theorem 4.3, Corollary 5.1, and (5.1), (5.2), we obtain

$$|H_{3,1}(f)|\le|a_3||H_{2,2}(f)|+|a_4||a_4-a_2a_3|+|a_5||H_{2,1}(f)|$$

$$\le\frac{1}{2}\cdot\frac{23}{48}+\frac{1}{4}\cdot\frac{11}{24}+\frac{1}{2}\cdot\frac{1}{2}=\frac{29}{48}.$$

Theorem 5.3. If f is of the form (1.1) belongs to $\mathcal{S}_{SC}^*(\sin z)$, then

$$|H_{4,1}(f)|\le\frac{26759147}{8294400}.$$

Proof: From (1.8) and using the triangle inequality, we obtain

$\left|H_{4,1}(f)\right| \leq |a_7|\left|H_{3,1}(f)\right| + |a_6||\rho_1| + |a_5||\rho_2| + |a_4||\rho_3|$.

We must find ρ_1, ρ_2 and ρ_3.

From (3.3), (3.4), (3.5), and (3.6) we will find ρ_1 as follows:

i. $a_2a_5 - a_3a_4 = \frac{1}{256}c_1c_4 - \frac{1}{384}{c_1}^3c_2 + \frac{1}{64}{c_2}^2c_1 - \frac{1}{64}{c_1}^2c_3 - \frac{1}{32}c_2c_3 + \frac{1}{768}{c_1}^5$

$$= \frac{1}{256}c_1(c_4 - 4c_1\,c_3) - \frac{1}{384}c_2({c_1}^3 - 6c_1\,c_2 + 12c_3) + \frac{1}{768}{c_1}^5.$$

Using the triangle inequality and Lemmas 2.1, 2.2, and 2.3, we obtain

$$\begin{aligned}|a_2a_5 - a_3a_4| &\leq \frac{1}{256}\cdot 2\cdot 2\cdot 7 + \frac{1}{384} \cdot 2(2|1| + 2|6 - 2(1)| + 2|1 - 6 + 12|) + \frac{1}{768}\cdot 32\\ &= \frac{7}{64} + \frac{1}{192}(2 + 8 + 14) + \frac{1}{24} = \frac{7}{64} + \frac{1}{8} + \frac{1}{24}\\ &= \frac{53}{192}. \quad (5.3)\end{aligned}$$

ii. $a_5 - a_2a_4 = \frac{1}{8}c_4 + \frac{9}{128}{c_1}^2c_2 - \frac{1}{32}{c_2}^2 - \frac{5}{32}c_1\,c_3 - \frac{1}{384}{c_1}^4$

$$= -\frac{1}{384}c_1({c_1}^3 - 27c_1\,c_2 + 60c_3) + \frac{1}{8}\left(c_4 - \frac{1}{4}{c_2}^2\right)$$

Using the triangle inequality and Lemmas 2.1, 2.2, and 2.3, we obtain

$$\begin{aligned}|a_5 - a_2a_4| &\leq \frac{1}{384}\cdot 2(2|1| + 2|27 - 2(1)| + 2|1 - 27 + 60|) + \frac{1}{8}\cdot 2\\ &= \frac{1}{192}(2 + 50 + 68) + \frac{1}{4} = \frac{5}{8} + \frac{1}{4} = \frac{7}{8}. \quad (5.4)\end{aligned}$$

From (3.3), (3.4), (3.5), and (3.6), we will find ρ_2.

$$\begin{aligned}a_3a_5 - a_4^2 &= \frac{1}{32}c_2c_4 + \frac{7}{1024}{c_1}^2{c_2}^2 - \frac{1}{128}{c_2}^3 - \frac{1}{128}c_1\,c_2c_3 - \frac{1}{64}{c_1}^2c_4\\ &\quad - \frac{1}{256}{c_1}^4c_2 + \frac{5}{384}{c_1}^3c_3 - \frac{1}{9216}{c_1}^6 - \frac{1}{64}{c_3}^2\\ &= \frac{1}{32}c_4\left(c_2 - \frac{1}{2}{c_1}^2\right) - \frac{1}{128}{c_2}^2\left(c_2 - \frac{7}{8}{c_1}^2\right)\\ &\quad + \frac{5}{384}c_3\left({c_1}^3 - \frac{3}{5}c_1\,c_2 - \frac{6}{5}c_3\right)\\ &\quad - \frac{1}{256}{c_1}^4\left(c_2 + \frac{1}{36}{c_1}^2\right).\end{aligned}$$

Using the triangle inequality and Lemmas 2.1, 2.3, and 2.4, we obtain

$$\begin{aligned}&|a_3a_5 - a_4^2|\\ &\leq \frac{1}{32}\cdot 2\cdot 2 + \frac{1}{128}\cdot 4\cdot 2 + \frac{5}{384}\cdot 2\left(2|1| + 2\left|\frac{3}{5} - 2(1)\right| + 2\left|1 - \frac{3}{5} - \frac{6}{5}\right|\right)\\ &+ \frac{1}{256}\cdot 16\cdot\frac{19}{9} = \frac{1}{8} + \frac{1}{16} + \frac{5}{192}\left(2 + \frac{22}{5}\right) + \frac{19}{144} = \frac{1}{8} + \frac{1}{16} + \frac{1}{6} + \frac{19}{144}\\ &= \frac{35}{72}. \quad (5.5)\end{aligned}$$

Using the triangle inequality and Theorem 3.1, Corollary 5.1, (5.3) and (5.4), then

$$\begin{aligned}|\rho_1| &= |a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6(a_3 - {a_2}^2)|\\ &\leq |a_3||a_2a_5 - a_3a_4| + |a_4||a_5 - a_2a_4|\\ &\quad + |a_6|\left|H_{2,1}(f)\right| \leq \frac{1}{2}\cdot\frac{53}{192} + \frac{1}{4}\cdot\frac{7}{8} + \frac{1009}{1440}\cdot\frac{1}{2}\\ &= \frac{53}{384} + \frac{7}{32} + \frac{1009}{2880} = \frac{4073}{5760}. \quad (5.6)\end{aligned}$$

Using the triangle inequality and Theorem 3.1, (5.2), (5.4), and (5.5), then

$$\begin{aligned}|\rho_2| &= |a_3(a_3a_5 - a_4^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3)|\\ &\leq |a_3||a_3a_5 - a_4^2| + |a_5||a_5 - a_2a_4|\\ &\quad + |a_6||a_4 - a_2a_3| \leq \frac{1}{2}\cdot\frac{35}{72} + \frac{1}{2}\cdot\frac{7}{8} + \frac{1009}{1440}\cdot\frac{11}{24}\\ &= \frac{35}{144} + \frac{7}{16} + \frac{11099}{34560} = \frac{34619}{34560}. \quad (5.7)\end{aligned}$$

Using the triangle inequality and Theorem 3.1, (5.1), (5.3), and (5.5), then

$$\begin{aligned}|\rho_3| &= |a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_2a_4 - {a_3}^2)|\\ &\leq |a_4||a_3a_5 - a_4^2| + |a_5||a_2a_5 - a_3a_4|\\ &\quad + |a_6||a_2a_4 - {a_3}^2| \leq \frac{1}{4}\cdot\frac{35}{72} + \frac{1}{2}\cdot\frac{53}{192} + \frac{1009}{1440}\cdot\frac{23}{48}\\ &= \frac{35}{288} + \frac{53}{384} + \frac{23207}{69120} = \frac{41147}{69120}. \quad (5.8)\end{aligned}$$

Using Theorem 3.1, (5.6), (5.7), and (5.8), we obtain

$$\begin{aligned}\left|H_{4,1}(f)\right| &\leq |a_7|\left|H_{3,1}(f)\right| + |a_6||\rho_1| + |a_5||\rho_2| + |a_4||\rho_3|\\ &\leq \frac{31}{9}\cdot\frac{29}{48} + \frac{1009}{1440}\cdot\frac{4073}{5760} + \frac{1}{2}\cdot\frac{34619}{34560} + \frac{1}{4}\cdot\frac{41147}{69120}\\ &= \frac{899}{432} + \frac{4109657}{8294400} + \frac{34619}{69120} + \frac{41147}{276480}\\ &= \frac{26759147}{8294400}.\end{aligned}$$

6. Conclusion

Several previous studies inspired this study. In this article, we have obtained the upper bounds of some coefficient problems for functions in the class $\mathcal{S}^*_{SC}(\sin z)$ including Taylor coefficients, logarithmic coefficients, and Hankel determinants of logarithmic coefficients. The results presented in this article may be the subject of further research on higher-order Hankel determinants of logarithmic coefficients and other coefficient problems, for instance, the Fekete-Szegö functional. Additionally, for another particular value of φ, several other classes of functions that are starlike with respect to symmetric conjugate points can also be studied.

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Conflict of Interest

The authors declare that there are no conflicts of interest.

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