Some characterizations of strongly π-regular rings

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Abstract :

A ring R is said to be strongly π-regular if for every a∈R there exist a positive integers n, b∈R such that an = an+1b. In this paper it has been proved that an abalian ring R is strongly π-regular if and only the set of all nilpotent element of R coincide with the Jacobson radical and R/J(R) is strongly-regular. Also we give some other characterizations of this kind of rings.

1. Introduction

Let R be a ring associative with unity. A ring R is said to be strongly π-regular if for each a in R there exist a positive integers n, $b \in R$ such that $a^n = a^{n+1} b$. By results of azumaya [3] and Deshinger [6], the element b can be chosen to commute with a. In particular; this definition is left– right symmetric. Recently, many authors have been studied strongly the π-regular rings from different view of points, see [2, 3, 6, 7, and 5]. In this paper, we shall prove that if R is an abelian ring, then R is strongly π-regular if and only if R/N(R) is strongly regular and N(R) = J(R). Also we shall give some characterizations of strongly π-regular ring.

The following notations will be used throughout this paper. We use "J(R)" to mean "the Jacobian radical", "N(R)" to mean "the set of all nilpotent elements of R", and "E(R)" to mean "the set of all idempotent elements of R'. The symbol "U(R)" is used to mean " the group of units", and the symbol " ● " is used to mean the end of proof.

Now, let R be a ring. An element $a \in R$ is said to be regular if there exists $b \in R$ such that a = aba. An element $a \in R$ is said to be strongly π-regular if there exists a positive integer n and $b \in R$ such that $a^n = a^{n+1} b$ and ab = ba. The ring R is said to be strongly π-regular if every element of R is strongly π-regular.

2. Basic properties of strongly π - regular rings

Definition (2-1)[9]. An element a in a ring R is said to be strongly regular if there exist c, and b in R such that $a^2 b = a = ca^2$.

Lemma (2.2) [9]. Let R be a ring .Then for any a in R, the following conditions are equivalent

1. a is strongly regular element.
2. a = aba and ab = ba for some $b \in R$.
3. a = aua and au = ua for some u in U(R).
4. a = ue = eu for some e in E(R) and for some u in U(R).

Lemma (2.3). Let R be a ring. Then for any a in R, the following statements are equivalent:

1. **a** is strongly π - regular element.
2. there exist a positive integers n , $e \in E(R)$ and $u \in E(R)$ such that

   $a^n = ea^n = eu$ and a, e, and u are commute.
3. There exists $e \in E(R)$ such that ea = ae is strongly regular and $a(1- e) \in N(R)$.

Proof. (1) ==➔ (2). Given (1) there exist a positive integers n**,** $b \in R$ such that $a^n = a^{n+1} b$ and ab = ba. Then we have $a^n = a^n (ab) = a^{n+1} b (ab) = a^{n+2} b^2 = \ldots\ldots = a^{2n} b^n = a^n b^n a^n$.

Set $e = a^n b^n$ and note that e is idempotent and $a^n = ea^n = a^n e$.

Now set $u = a^n + 1- e$ and $v = e b^n +1-e$ and then note that uv = vu =1. Moreover $eu = e (a^n+1- e) = ea^n = a^n$ similarly $ue = a^n$ and hence $a^n = eu$

= ue. Finally since ea = ae, then au = ua.

(2) ==➔ (3). From (2), we have a^n = ea^n = eu and a, e, and u are commute. Set w = a^{n-1} u^{-1}. Then ea (w) ea = ea and (ea)w = w(ea) thus, by lemma (2.2), we have ea is strongly regular in R. By noting that a(1-e) = (1-e)a , since ae = ae ,we have

$(a(1- e))^n = a^n(1- e) = (1- e)a^n = (1- e)eu = 0$, that is a(1- e) = (1- e)a$\in$ N(R)

(3) ==➔ (1). From (3) we have $a (1-e))^n = 0$ which implies that $a^n = a^ne$. Since ea is strongly regular, then by (lemma 2.2), there exist c$\in$ E(R) and v$\in$ U(R) such that ea = cv = vc.

Now $a^n = ea^n = (ea)^n = (cv)^n = cv^n$.

Set $b = cv^{-m}$, then $a^nba^n = a^n$ and $a^nb = ba^n$ .That is a is strongly π-regular in R. ●

3. The Main results

This section is devoted for introducing the main results that we are obtained in this research. One of them is the following theorem, it is a generalization of the proposition (12) found in [10].

Theorem (3.1). For any ring R, the following statements are equivalent:

1. R is strongly π-regular**.**
2. For each a$\in$R there exist e$\in$ E(R), u$\in$U(R) and w$\in$N(R) such that a= cu + w and a, c, and u are commute.

proof. (1) ==➔ (2). Let a be in R. Since R is strongly π-regular, then by lemma (2.3), there exists $e\in E(R)$ such that ea = ae is strongly regular and $a(1-e) = (1-e)a\in N(R)$ and lemma (2.2) implies that ea = cu for some $c\in E(R)$ and $u\in U(R)$.

Now set w = a (1- e), and note that ea + w = ea +a (1 - e) = a. Thus a = ea + w = cu + w. Obviously e, w, and u are commute.

(2) ==➔ (1). Let $a\in R$ such that a = cu +w for some $c\in E(R)$, $u\in U(R)$, and $w\in N(R)$ and c, u, and w are commute.

Set $b=u^{-1}$ then we note that b commute with each of a, c and w. Now $a-a^2b = cu + w (cu + w)^2u^{-1} = w - w(2c + wu^{-1})$. That is $a - a^2 b$ is a nilpotent element of R and $au^{-1} = u^{-1} a$, hence a is strongly π-regular. ●

If R is non-commutative ring, then N(R) needs not to be left or right ideal of R. We shall prove that if R is abelian ring (in which every idempotent is central), then R is strongly π-regular if and only if N(R) = J(R) and R/J(R) are strongly regular rings. This main result is the subject of the next theorem, but first we need the following lemma.

Lemma (3.2). [1, prop.5.3]. Let R be an abelian ring and R/I is regular for some nil ideal I of R and let ab + I is an idempotent element in R/I. Then there exists an idempotent element e in R such that e belong to the subring [ab] generated by the element ab.

Theorem (3.3). Let R be an abelian ring. Then the following are equivalent

1. R is strongly π-regular.

2. N(R) = J(R) and R/J (R) is strongly regular.

Proof. (1) ==➔ (2). Let $0 \neq a \in N(R)$ with $a^n = o$, for some positive integer n, and consider r a, where $r \in R$, then there exist $e \in E(R)$ and $u \in U(R)$ such that ra = eu = ue. Since R is abelian , then $(ea)^n = e\, a^n = 0$. Now $0 = r\,(ea)^n = r\, e\, a\, (ea)^{n-1} = e(ra)\,(ea)^{n-1} = e\, u\,(ea)^{n-1} = u(ea)^{n-1}$. This implies that $(ea)^{n-1} = 0$ .By repeating this process (n - 1) times, we get ea = 0. But $(1-e)\, ra \in N(R)$ (by lemma 2.3), and from other hand (1- e) ra = ra – era = ra – r (ea) = r a. Thus $ra = (1-e)ra \in N(R)$ for all $r \in R$, similarly $ar \in N(R)$ for all $r \in R$. consequently aR and Ra are nil and so they are contained in J(R) , hence we have $N(R) \subseteq J(R)$. We also get $J(R) \subseteq N(R)$ since R is strongly π-regular, showing N(R) = J(R) it then follows R/J(R) is reduced to strongly π-regular and hence it's strongly regular.

(2) ==➔ (1). Assume that R/N(R) is strongly regular and N(R) = J(R).

Let $a \in R$, so $a + N \in R/N(R)$.

Since R/N(R) is regular, then a + N(R) = (a + N(R)) (b + N (R)) (a + N(R)) for some $b + N \in R/N$.

This implies a + N(R) = (ab + N(R))(a + N(R)). We observe that (ab + N(R)) is idempotent in R/ N(R). By lemma. (3.2) there exists an idempotent $e \in [ab]$ such that (ab + N(R)) = e + N(R). These implies $ab - e \in N(R)$ and hence $aba - ea \in N(R)$. Thus $a - ea = a - ae \in N(R)$.Then there exists a positive integer m such that $(a^m(1-e)^m) = 0$, which implies $a^m = a^m e$. But $e \in [ab]$, then we have $a^m = a^m abc$ for some $c \in R$. Hence $a^m = a^{m+1}d$, where d = bc. Therefore R is strongly π-regular. ●

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