ON R^h-TRIRECURRENT FINSLER SPACES

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Abstract:

The concept of *recurrent curvature of an n-dimensional Riemannian space* was extended to a Finsler space by A. Moór ([7],[8],[9]). Shalini Dikshit [3] defined the *birecurrent Finsler space* and F.Y.A.Qasem [10] defined *the generalized birecurrent Finsler space and their properties* considering the Cartan's curvature tensor $R^i_{jkh}(x,y),(y=\dot{x})$.The object of the present paper is to study *trirecurrent Finsler space* considering the Cartan's curvature tensor $R^i_{jkh}(x,y)$.

1. INTRODUCTION

Let $F_n = (M_n, F(x,y))$ be a Finsler space on a differentiable space M_n, equipped with the fundamental function $F(x,y)$. We denote the fundamental metric tensor

$$(1.1) \qquad g_{ij}(x,y)\frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x,y), \dot{\partial}_i := \frac{\partial}{\partial y^i} .$$

The *(h)hv*-torsion tensor C_{ijk} defined by Makoto Matsumoto[4]

$$(1.2) \qquad C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k F^2 .$$

This tensor satisfies the following identities

$$(1.3) \qquad \text{(a)} \quad C_{ijk}y^i = C_{kij}y^i = C_{jki}y^i = 0 ,$$

$$\text{(b)} \quad C^i_{jk}y^j = C^i_{kj}y^j = 0 .$$

The *(v)hv*-torsion tensor C^i_{jk} is the associate tensor of C_{ijk} and is defined by

$$\text{(c)} \quad C^h_{ik} := g^{hj}C_{ijk} .$$

The unit vector l^i in the direction of y^i is given by

$$(1.4) \qquad l^i := \frac{y^i}{F} .$$

For an arbitrary vector field X^i, Cartan deduced ([1],[2])

$$(1.5) \qquad X^i_{|k} := \partial_k X^i - (\dot{\partial}_r X^i) G^r_k + X^r \Gamma^{*i}_{rk} .$$

The function Γ^{*i}_{rk} and G^r_k are defined by

$$(1.6) \qquad \text{(a)} \quad \Gamma^{*i}_{rk} := \Gamma^{i}_{rk} - C^i_{mr}\Gamma^{m}_{sk} y^s ,$$

$$\text{(b)} \quad G^r_k = \Gamma^{*r}_{sk} y^s .$$

The process of covariant differentiation known as *h*-covariant differentiation (Cartan's second kind covariant differentiation) . Makoto Matsumoto ([4],[5]) calls this derivative as *h*-covariant derivative.

The metric tensor g_{ij} and the associative metric tensor g^{ij} are covariant constant with respect to the above process, i.e.

(1.7) (a) $g_{ij|k} = 0$ and (b) $g^{ij}_{|k} = 0$.

The *h*-covariant derivative of the vector y^i vanish identically, i.e.

(1.8) $y^i_{|k} = 0$.

The process of *h*-covariant differentiation defined above commute with the partial differentiation with respect to y^j according to

(1.9) $\dot{\partial}_j \left(X^i_{|k} \right) - \left(\dot{\partial}_j X^i \right)_{|k} = X^r \left(\dot{\partial}_j \Gamma^{*i}_{rk} \right) - \left(\dot{\partial}_r X^i \right) P^r_{jk}$.

The Cartan curvature tensor k^i_{rhk} defined by

(1.10) (a) $k^i_{rhk} := \partial_k \Gamma^{*i}_{hr} + \dot{\partial}_l \Gamma^{*i}_{rk} G^l_h + \Gamma^{*i}_{mk} \Gamma^{*m}_{hr} - k/h$ * .

This tensor satisfies the following identity (which is one of Bianchi identities)

(b) $k^i_{jkh} + k^i_{hjk} + k^i_{khj} = 0.$

Also the above tensor satisfies the following relation

(1.11) $k^i_{jkh} y^j = H^i_{kh}$,

where

* *k/h* means the subtraction from the former term by interchanging the indices *k* and *h*

(1.12) $$H^{i}_{kh} \coloneqq \partial_h G^i_k + G^r_k G^i_{rh} - h/k \ .$$

Matsumoto Makoto calls it *h(v)*-torsion tensor.This tensor is positively homogeneous of degree one in y^i

The *h*-curvature tensor R^i_{jkh} (Cartan's third curvature tensor) defined by

(1.13)

$$R^{i}_{jkh} = \partial_h \Gamma^{*i}_{jk} + \left(\dot{\partial}_l \Gamma^{*i}_{jk}\right)\Gamma^{*l}_{sh} y^s + C^i_{jm}\left(\partial_k \Gamma^{*m}_{sh} y^s - \Gamma^{*m}_{kl}\Gamma^{l}_{sh} y^s\right) + \Gamma^{*i}_{mk}\Gamma^{m}_{jh} - h/k$$

.

This tensor satisfies the following identity

(1.14)

$$R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + y^m\left(R^l_{mkh}P^r_{ijl} + R^l_{mjk}P^r_{ihl} + R^l_{mhj}P^r_{ikl}\right) = 0,$$

where P^i_{jkh} is defined by

(1.15) (a) $P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jm}P^m_{kh} - C^i_{jh|k}$,

which satisfies

(b) $P^i_{jkh} y^j = \Gamma^{*i}_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r$.

The tensor R^i_{jkh} satisfies the following relations

(1.16) (a) $R^i_{jkh} = k^i_{jkh} + C^i_{jm}k^m_{rkh}y^r$,

(b) $R^i_{jkh} y^j = k^i_{jkh} y^j = H^i_{kh}$.

The associate tensor R_{ijhk} of R^r_{ihk} is given by

(1.17) $R_{ijhk} = g_{jr} R^r_{ihk}$.

The curvature tensor H^i_{jkh} defined by

(1.18) $$H^i_{jkh} \coloneqq \partial_h G^i_{jk} + G^r_{jk}G^i_{rh} + G^i_{rjh}G^r_k - h/k \ .$$

Makoto Matsumoto calls it *h*-curvature tensor. However we shall call it the *h*-curvature tensor of Berwald . This tensor is positively homogeneous of degree zero in y^i.The curvature tensor H^i_{jkh} and the tensor H^i_{jk} are related by

(1.19) (a) $H^i_{jkh} y^j = H^i_{kh}$ and (b) $H^i_{jkh} = \dot{\partial}_j H^i_{kh}$.

The deviation tensor H^i_h ,given by

(1.20) $$H^i_h := 2\partial_h G^i - \partial_r G^i_h y^r + 2G^i_{hs} G^s - G^i_s G^s_h ,$$

is positively homogeneous of degree two in y^i. Berwald constructed the tensor H^i_{jk} from the deviation tensor H^i_h according to

(1.21) $$H^i_{kh} = \frac{1}{3}\left(\dot{\partial}_h H^i_k - \dot{\partial}_k H^i_h\right) .$$

In view of Euler's theorem on homogeneous functions we have the following relations

(1.22) $$H^i_{jk} y^j = -H^i_{kj} y^j = H^i_k .$$

Also the above tensor satisfies the following

(1.23) (a) $H_k := H^r_{kr}$, (b) $H := \frac{1}{n-1} H^r_r$ and (c) $y_i H^i_{kh} = 0$.

The tensor $H_{.jkh}$ defined by

(1.24) $$H_{.jkh} := g_{ik} H^i_{jh} .$$

For a recurrent [12] and birecurrent [3] Finsler spaces we have respectively

(1.25) $$R^i_{jkh|m} = \lambda_m R^i_{jkh} , \quad R^i_{jkh} \neq 0$$

and

(1.26) $$R^i_{jkh|m|l} = a_{lm} R^i_{jkh} , \quad R^i_{jkh} \neq 0 ,$$

where the non-zero covariant vector field λ_m being the recurrence vector field and a_{lm} is a non-zero covariant tensor field of order 2 called birecurrence tensor field .

2. R^h-TRIRECURRENT SPACE

Let us consider a Finsler space whose Cartan third curvature tensor satisfies

(2.1) $$R^i_{jkh|m|l|n} = b_{n1m} R^i_{jkh} \quad , \quad R^i_{jkh} \neq 0 \ ,$$

where b_{n1m} is a non-zero covariant tensor field of order 3 called as trirecurrence tensor field . The space satisfying the condition (2.1) will be called *an* R^h*-trirecurrent space* .

Let us consider an R^h-birecurrent space F_n satisfies (1.26) .Differentiating (1.26) covariantly with respect to Cartan's connection Γ^{*i}_{kh} we have (2.1) where $b_{nlm} = \left(a_{lm|n} + a_{lm}\lambda_n\right)$. This shows that the space considered is an R^h-trirecurrent . *Therefore we conclude that every* R^h*-birecurrent space is* R^h*-trirecurrent* .

Now we shall consider an R^h-trirecurrent space characterized by (2.1) . Transvecting (2.1) by g_{ip} and using the fact that the metric tensor is covariant constant with respect to Cartan's connection Γ^{*i}_{kh} , we have

(2.2) $$R_{jpkh|m|l|n} = b_{nlm} R_{jpkh} \ .$$

The contraction of the indices i and h in (2.1) gives

(2.3) $$R_{jk|m|l|n} = b_{nlm} R_{jk} \ ,$$

showing that the Ricci tensor R_{jk} of an R^{h}-trirecurrent space is trirecurrent . A space whose Ricci tensor is trirecurrent with respect to Cartan's connection may be called as Ricci trirecurrent space .

Thus we may conclude

Theorem 2.1 : *An* R^{h}*-trirecurrent space is always a Ricci trirecurrent space.*

We know that the curvature tensor R_{ijkh} of a three dimensional Finsler space is of the form [6]

(2.4) $$R_{ijkh} = g_{ik} L_{jh} + g_{jh} L_{ik} - k / h \ ,$$

where

(2.5) $$L_{ik} = \frac{1}{n-2}\left(R_{ik} - \frac{r}{2} g_{ik} \right),$$

(2.6) $$r = \frac{1}{n-1} R_i^i \ .$$

Transvecting (2.3) by g^{jp} , we have

(2.7) $$R^{p}_{k|m|l|n} = b_{nlm} R^{p}_{k} \ .$$

If we contract the indices p and k in this equation ,we find

(2.8) $$r_{|m|l|n} = b_{nlm} r \ .$$

In view of this equation and the equation (2.3) the third covariant differentiation of (2.5) gives

(2.9) $$L_{ik|m|l|n} = b_{nlm} L_{ik} \ .$$

Differentiating (2.4) covariantly three times with respect to x^m, x^l and x^n successively in the sense of Cartan and using the equations (2.9) and $g^{rj} R_{ijkh} = R^{r}_{ikh}$ imply (2.1) .

Thus we may conclude

Theorem 2.2 : *The three dimensional Ricci trirecurrent space is necessarily an* $\mathcal{R}^h$*-trirecurrent space .*

Makoto Matsumoto [6] introduced a Finsler space F_n $(n > 3)$ for which the tensor R_{ijkh} satisfies the condition (2.4) and called it R3-like Finsler space.

If we consider an R3-like Ricci trirecurrent Finsler space then adopting the same process as in theorem (2.2) we may show that it is an $\mathcal{R}^h$-trirecurrent space.

Thus we may conclude

Theorem 2.3 : *An R3-like Ricci trirecurrent space is an* $\mathcal{R}^h$*-trirecurrent space .*

Transvecting (2.1) by y^j and using (1.16b) ,we get

(2.10) $$H^i_{kh|m|l|n} = b_{nlm} H^i_{kh} \quad .$$

Further transvection of (2.10) by y^k and using (1.22), we get

(2.11) $$H^i_{h|m|l|n} = b_{nlm} H^i_h \quad .$$

Contracting the indices *i* and *h* in (2.11) and using (1.23b) ,we get

(2.12) $$H_{|m|l|n} = b_{nlm} H \quad .$$

Contracting the indices *i* and *h* in (2.10) and using (1.23a), we get

(2.13) $$H_{k|m|l|n} = b_{nlm} H_k \quad .$$

Thus we may conclude the above discussion as follows

Theorem 2.4 : *The tensors* H^i_{kh}, H^i_h *,the vector* H_k *and the scalar H of an* $\mathcal{R}^h$*-trirecurrent space are all h-trirecurrent .*

Now we shall try to find the necessary and sufficient condition for Berwald curvature tensor H^i_{jkh} to be *h*-trirecurrent .

For this purpose we differentiate (2.10) partially with respect to y^j, we get

(2.14) $$\dot\partial_j H^i_{kh|m|l|n} = (\dot\partial_j b_{nlm}) H^i_{kh} + b_{nlm} H^i_{jkh} \; ,$$

where $\dot\partial_j H^i_{kh} = H^i_{jkh}$. Using the commutation formula(1.9) three times in (2.14)

where $P^r_{jk} := (\dot\partial_j \Gamma^{*r}_{hk}) y^h$,we get

(2.15)

$$\begin{aligned} & H^i_{jkh|m|l|n} + [\{H^r_{kh}\dot\partial_j\Gamma^{*i}_{rm} - H^i_{rh}\dot\partial_j\Gamma^{*r}_{km} - H^i_{kr}\dot\partial_j\Gamma^{*r}_{hm} - H^i_{rkh}P^r_{jm}\}_{|l} + H^r_{kh|m}\dot\partial_j\Gamma^{*i}_{rl} - H^i_{rh|m}\dot\partial_j\Gamma^{*r}_{kl} \\ & - H^i_{kr|m}\dot\partial_j\Gamma^{*r}_{hl} - H^i_{kh|r}\dot\partial_j\Gamma^{*r}_{ml} - \{H^i_{rkh|m} + H^s_{kh}\dot\partial_r\Gamma^{*i}_{sm} - H^i_{sh}\dot\partial_r\Gamma^{*s}_{km} - H^i_{ks}\dot\partial_r\Gamma^{*s}_{hm} - H^i_{skh}P^s_{rm}\}P^r_{jl}]_{|n} \\ & + H^r_{kh|m|l}\dot\partial_j\Gamma^{*i}_{rn} - H^i_{rh|m|l}\dot\partial_j\Gamma^{*r}_{kn} - H^i_{kr|m|l}\dot\partial_j\Gamma^{*r}_{hn} - H^i_{kh|r|l}\dot\partial_j\Gamma^{*r}_{mn} - H^i_{kh|m|r}\dot\partial_j\Gamma^{*r}_{ln} - [\{H^i_{rkh|m} \\ & + H^s_{kh}\dot\partial_r\Gamma^{*i}_{sm} - H^i_{sh}\dot\partial_r\Gamma^{*s}_{km} - H^i_{ks}\dot\partial_r\Gamma^{*s}_{hm} - H^i_{skh}P^s_{rm}\}_{|l} + H^s_{kh|m}\dot\partial_r\Gamma^{*i}_{sl} - H^i_{sh|m}\dot\partial_r\Gamma^{*s}_{kl} \\ & - H^i_{ks|m}\dot\partial_r\Gamma^{*s}_{hl} - H^i_{kh|s}\dot\partial_r\Gamma^{*s}_{ml} - \{H^i_{skh|m} + H^t_{kh}\dot\partial_s\Gamma^{*i}_{tm} - H^i_{th}\dot\partial_s\Gamma^{*t}_{km} - H^i_{kt}\dot\partial_s\Gamma^{*t}_{hm} - H^i_{tkh}P^t_{sm}\}P^s_{rl}]P^r_{jn} \\ & = (\dot\partial_j b_{nlm}) H^i_{kh} + b_{nlm} H^i_{jkh} \end{aligned}$$

This equation shows that

(2.16) $$H^i_{jkh|m|l|n} = b_{nlm} H^i_{jkh}$$

if and only if the following equation holds, i.e.

(2.17)

$$\begin{aligned} & [\{H^r_{kh}\dot\partial_j\Gamma^{*i}_{rm} - H^i_{rh}\dot\partial_j\Gamma^{*r}_{km} - H^i_{kr}\dot\partial_j\Gamma^{*r}_{hm} - H^i_{rkh}P^r_{jm}\}_{|l} + H^r_{kh|m}\dot\partial_j\Gamma^{*i}_{rl} - H^i_{rh|m}\dot\partial_j\Gamma^{*r}_{kl} - H^i_{kr|m}\dot\partial_j\Gamma^{*r}_{hl} \\ & - H^i_{kh|r}\dot\partial_j\Gamma^{*r}_{ml} - \{H^i_{rkh|m} + H^s_{kh}\dot\partial_r\Gamma^{*i}_{sm} - H^i_{sh}\dot\partial_r\Gamma^{*s}_{km} - H^i_{ks}\dot\partial_r\Gamma^{*s}_{hm} - H^i_{skh}P^s_{rm}\}P^r_{jl}]_{|n} + H^r_{kh|m|l}\dot\partial_j\Gamma^{*i}_{rn} \\ & - H^i_{rh|m|l}\dot\partial_j\Gamma^{*r}_{kn} - H^i_{kr|m|l}\dot\partial_j\Gamma^{*r}_{hn} - H^i_{kh|r|l}\dot\partial_j\Gamma^{*r}_{mn} - H^i_{kh|m|r}\dot\partial_j\Gamma^{*r}_{ln} - [\{H^i_{rkh|m} + H^s_{kh}\dot\partial_r\Gamma^{*i}_{sm} - H^i_{sh}\dot\partial_r\Gamma^{*s}_{km} \\ & - H^i_{ks}\dot\partial_r\Gamma^{*s}_{hm} - H^i_{skh}P^s_{rm}\}_{|l} + H^s_{kh|m}\dot\partial_r\Gamma^{*i}_{sl} - H^i_{sh|m}\dot\partial_r\Gamma^{*s}_{kl} - H^i_{ks|m}\dot\partial_r\Gamma^{*s}_{hl} - H^i_{kh|s}\dot\partial_r\Gamma^{*s}_{ml} - \{H^i_{skh|m} + H^t_{kh}\dot\partial_s\Gamma^{*i}_{tm} \\ & - H^i_{th}\dot\partial_s\Gamma^{*t}_{km} - H^i_{kt}\dot\partial_s\Gamma^{*t}_{hm} - H^i_{tkh}P^t_{sm}\}P^s_{rl}]P^r_{jn} = (\dot\partial_j b_{nlm}) H^i_{kh} \end{aligned}$$

Thus the above discussion can be concluded as follows

Theorem 2.5 : *The Berwald's curvature tensor* H^i_{jkh} *of an* R^h*-trirecurrent space is h-trirecurrent if and only if the equation (2.17) holds .*

The curvature tensor R^i_{jkh} and k^i_{jkh} are connected by the formula [11]

(2.18) $$R^i_{jkh} = k^i_{jkh} + C^i_{jr} H^r_{kh} \ .$$

Differentiating (2.18) covariantly with respect to x^m, we get

(2.19) $$R^i_{jkh|m} = k^i_{jkh|m} + C^i_{jr|m} H^r_{kh} + C^i_{jr} H^r_{kh|m} \ .$$

Again differentiating (2.19) covariantly with respect to x^l, we get

(2.20)

$$R^i_{jkh|m|l} = k^i_{jkh|m|l} + C^i_{jr|m|l} H^r_{kh} + C^i_{jr|m} H^r_{kh|l} + C^i_{jr|l} H^r_{kh|m} + C^i_{jr} H^r_{kh|m|l} \ .$$

Again differentiating (2.20) covariantly with respect to x^n, we get

(2.21)

$$R^i_{jkh|m|l|n} = k^i_{jkh|m|l|n} + C^i_{jr|m|l|n} H^r_{kh} + C^i_{jr|m|l} H^r_{kh|n} + C^i_{jr|m|n} H^r_{kh|l} + C^i_{jr|m} H^r_{kh|l|n} + C^i_{jr|l|n} H^r_{kh|m} + C^i_{jr|l} H^r_{kh|m|n} + C^i_{jr|n} H^r_{kh|m|l} + C^i_{jr} H^r_{kh|m|l|n}$$

,

using (2.1) ,(2.18) and (2.10) in (2.21) ,we get

(2.22)

$$b_{nlm} k^i_{jkh} = k^i_{jkh|m|l|n} + C^i_{jr|m|l|n} H^r_{kh} + C^i_{jr|m|l} H^r_{kh|n} + C^i_{jr|m|n} H^r_{kh|l} + C^i_{jr|m} H^r_{kh|l|n} + C^i_{jr|l|n} H^r_{kh|m} + C^i_{jr|l} H^r_{kh|m|n} + C^i_{jr|n} H^r_{kh|m|l}$$

.

From (2.22) we may conclude that the curvature tensor k^i_{jkh} is *h*-trirecurrent in an R^h-trirecurrent space if and only if

(2.23)
$$C^i_{jr|m|l|n}H^r_{kh}+C^i_{jr|m|l}H^r_{kh|n}+C^i_{jr|m|n}H^r_{kh|l}+C^i_{jr|m}H^r_{kh|l|n}$$
$$+C^i_{jr|l|n}H^r_{kh|m}+C^i_{jr|l}H^r_{kh|m|n}+C^i_{jr|n}H^r_{kh|m|l}=0$$

Thus , we have

Theorem 2.6 : Cartan *curvature tensor* k^i_{jkh} *of an* R^h-*trirecurrent space is h-trirecurrent if and only if the condition (2.23) is satisfied .*

3. CERTAIN IDENTITIES

We know that the tensor R_{ijhk} satisfies the identity ([1],[2])

(3.1)
$$R_{ijhk}+R_{ihkj}+R_{ikjh}+(C_{ijr}k^r_{shk}+C_{ihr}k^r_{skj}+C_{ikr}k^r_{sjh})y^s=o,$$

which in view of (1.16b) reduces to

(3.2)
$$R_{ijhk}+R_{ihkj}+R_{ikjh}+(C_{ijr}H^r_{hk}+C_{ihr}H^r_{kj}+C_{ikr}H^r_{jh})=o.$$

Differentiating (3.2) covariantly with respect to x^m, we get

(3.3)
$$R_{ijhk|m}+R_{ihkj|m}+R_{ikjh|m}+(C_{ijr}H^r_{hk}+C_{ihr}H^r_{kj}+C_{ikr}H^r_{jh})_{|m}=o.$$

Again differentiating (3.3) covariantly with respect to x^l, we get

(3.4)
$$R_{ijhk|m|l}+R_{ihkj|m|l}+R_{ikjh|m|l}+(C_{ijr}H^r_{hk}+C_{ihr}H^r_{kj}+C_{ikr}H^r_{jh})_{|m|l}=o.$$

Again differentiating (3.4) covariantly with respect to x^n and using (2.2), we get

(3.5)
$$b_{nlm}(R_{ijhk}+R_{ihkj}+R_{ikjh})+(C_{ijr}H^r_{hk}+C_{ihr}H^r_{kj}+C_{ikr}H^r_{jh})_{|m|l|n}=o.$$

Using (3.2) in (3.5), we get

(3.6)

$$(C_{ijr}H^{r}_{hk} + C_{ihr}H^{r}_{kj} + C_{ikr}H^{r}_{jh})_{|m|l|n} - b_{nlm}(C_{ijr}H^{r}_{hk} + C_{ihr}H^{r}_{kj} + C_{ikr}H^{r}_{jh}) = o.$$

Transvecting (3.6) by y^{j} and using (1.22) and (1.3a), we get

(3.7) $$(C_{ikr}H^{r}_{h} - C_{ihr}H^{r}_{k})_{|m|l|n} = b_{nlm}(C_{ikr}H^{r}_{h} - C_{ihr}H^{r}_{k}) \ .$$

Transvecting (3.7) by g^{ji} and using the symmetric property of *(h)hv*-torsion tensor C_{ijk} in all its lower indices and using (1.3c) ,we get

(3.8) $$(C^{j}_{kr}H^{r}_{h} - C^{j}_{hr}H^{r}_{k})_{|m|l|n} = b_{nlm}(C^{j}_{kr}H^{r}_{h} - C^{j}_{hr}H^{r}_{k}) \ .$$

In view of (1.3c) and (1.17), the equation (3.4) can be written as

(3.9)

$$R^{i}_{jhk|m|l} + R^{i}_{hkj|m|l} + R^{i}_{kjh|m|l} + (C^{i}_{jr}H^{r}_{hk} + C^{i}_{hr}H^{r}_{kj} + C^{i}_{kr}H^{r}_{jh})_{|m|l} = o.$$

Using (1.16a) and (1.16b) in (3.9), we get

(3.10)

$$(k^{i}_{jhk} + k^{i}_{hkj} + k^{i}_{kjh})_{|m|l} + 2(C^{i}_{jr}H^{r}_{hk} + C^{i}_{hr}H^{r}_{kj} + C^{i}_{kr}H^{r}_{jh})_{|m|l} = o \ .$$

Using (1.10b) in (3.10) ,we get

(3.11) $$(C^{i}_{jr}H^{r}_{hk} + C^{i}_{hr}H^{r}_{kj} + C^{i}_{kr}H^{r}_{jh})_{|m|l} = o \ ,$$

which implies

(3.12)

$$C^{i}_{jr|m|l}H^{r}_{hk} + C^{i}_{jr|m}H^{r}_{hk|l} + C^{i}_{jr|l}H^{r}_{hk|m} + C^{i}_{jr}H^{r}_{hk|m|l} + C^{i}_{hr|m|l}H^{r}_{kj} + C^{i}_{hr|m}H^{r}_{kj|l}$$
$$+ C^{i}_{hr|l}H^{r}_{kj|m} + C^{i}_{hr}H^{r}_{kj|m|l} + C^{i}_{kr|m|l}H^{r}_{jh} + C^{i}_{kr|m}H^{r}_{jh|l} + C^{i}_{kr|l}H^{r}_{jh|m} + C^{i}_{kr}H^{r}_{jh|m|l} = o.$$

Differentiating (3.12) covariantly with respect to x^{n} in the sense of Cartan , we get

(3.13)

$$C^i_{jr|\mathrm{m}|l|n}H^r_{hk}+C^i_{jr|\mathrm{m}|l}H^r_{hk|n}+C^i_{jr|\mathrm{m}|n}H^r_{hk|l}+C^i_{jr|\mathrm{m}}H^r_{hk|l|n}+C^i_{jr|l|n}H^r_{hk|\mathrm{m}}+C^i_{jr|l}H^r_{hk|\mathrm{m}|n}$$
$$+C^i_{jr|n}H^r_{hk|\mathrm{m}|l}+C^i_{jr}H^r_{hk|\mathrm{m}|l|n}+C^i_{hr|\mathrm{m}|l|n}H^r_{kj}+C^i_{hr|\mathrm{m}|l}H^r_{kj|n}+C^i_{hr|\mathrm{m}|n}H^r_{kj|l}+C^i_{hr|\mathrm{m}}H^r_{kj|l|n}$$
$$+C^i_{hr|l|n}H^r_{kj|\mathrm{m}}+C^i_{hr|l}H^r_{kj|\mathrm{m}|n}+C^i_{hr|n}H^r_{kj|\mathrm{m}|l}+C^i_{hr}H^r_{kj|\mathrm{m}|l|n}+C^i_{kr|\mathrm{m}|l|n}H^r_{jh}+C^i_{kr|\mathrm{m}|l}H^r_{jh|n}$$
$$+C^i_{kr|\mathrm{m}|n}H^r_{jh|l}+C^i_{kr|\mathrm{m}}H^r_{jh|l|n}+C^i_{kr|l|n}H^r_{jh|\mathrm{m}}+C^i_{kr|l}H^r_{jh|\mathrm{m}|n}+C^i_{kr|n}H^r_{jh|\mathrm{m}|l}+C^i_{kr}H^r_{jh|\mathrm{m}|l|n}=o.$$

Multiplying (3.13) by y^m and using (1.15b) , we get

(3.14)

$$P^i_{jr|l|n}H^r_{hk}+P^i_{jr|l}H^r_{hk|n}+P^i_{jr|n}H^r_{hk|l}+P^i_{jr}H^r_{hk|l|n}+C^i_{jr|l|n}H^r_{hk|\mathrm{m}}y^m+C^i_{jr|l}H^r_{hk|\mathrm{m}|n}y^m$$
$$+C^i_{jr|n}H^r_{hk|\mathrm{m}|l}y^m+C^i_{jr}H^r_{hk|\mathrm{m}|l|n}y^m+P^i_{hr|l|n}H^r_{kj}+P^i_{hr|l}H^r_{kj|n}+P^i_{hr|n}H^r_{kj|l}+P^i_{hr}H^r_{kj|l|n}$$
$$+C^i_{hr|l|n}H^r_{kj|\mathrm{m}}y^m+C^i_{hr|l}H^r_{kj|\mathrm{m}|n}y^m+C^i_{hr|n}H^r_{kj|\mathrm{m}|l}y^m+C^i_{hr}H^r_{kj|\mathrm{m}|l|n}y^m+P^i_{kr|l|n}H^r_{jh}$$
$$+P^i_{kr|l}H^r_{jh|n}+P^i_{kr|n}H^r_{jh|l}+P^i_{kr}H^r_{jh|l|n}+C^i_{kr|l|n}H^r_{jh|\mathrm{m}}y^m+C^i_{kr|l}H^r_{jh|\mathrm{m}|n}y^m+C^i_{kr|n}H^r_{jh|\mathrm{m}|l}y^m$$
$$+C^i_{kr}H^r_{jh|\mathrm{m}|l|n}y^m=o.$$

Multiplying (3.14) by y^h and using (1.3b), (1.22) and

$P^i_{hk}y^h=0$, we get

(3.15)

$$(P^i_{jr}H^r_k-P^i_{kr}H^r_j)_{|l|n}+(C^i_{jr}H^r_{k|\mathrm{m}}-C^i_{kr}H^r_{j|\mathrm{m}})_{|l|n}\,y^m=0.$$

Consider the Bianchi identity for the tensor R^i_{jkh} given by [11]

(3.16)

$$R^i_{jkh|m}+R^i_{jmk|h}+R^i_{jhm|k}+y^s\left(R^r_{shm}P^i_{jkr}+R^r_{skh}P^i_{jmr}+R^r_{smk}P^i_{jhr}\right)=0,$$

where

(3.17) $$P^i_{jkh}=\dot\partial_h\Gamma^{*i}_{jk}-C^i_{jh|k}+C^i_{jr}C^r_{kh|s}y^s.$$

Here we should note that due to difference in the definition of tensor P^i_{jkh} , the form of Bianchi identity (3.16) differs from (4.3.3) of [11] .

Differentiating (3.16) covariantly with respect to x^l gives

(3.18)

$$R^i_{jkh|m|l} + R^i_{jmk|h|l} + R^i_{jhm|k|l} + y^s \left(R^r_{shm} P^i_{jkr} + R^r_{skh} P^i_{jmr} + R^r_{smk} P^i_{jhr} \right)_{|l} = 0 \; .$$

Differentiating (3.18) covariantly with respect to x^n gives

(3.19)

$$R^i_{jkh|m|l|n} + R^i_{jmk|h|l|n} + R^i_{jhm|k|l|n} + y^s \left(R^r_{shm} P^i_{jkr} + R^r_{skh} P^i_{jmr} + R^r_{smk} P^i_{jhr} \right)_{|l|n} = 0 \, .$$

In view of the condition (2.1) characterizing the R^h-trirecurrent space and the equation (1.16b), the identity (3.19) may be written as

(3.20)

$$b_{nlm} R^i_{jkh} + b_{nlh} R^i_{jmk} + b_{nlk} R^i_{jhm} + \left(H^r_{hm} P^i_{jkr} + H^r_{kh} P^i_{jmr} + H^r_{mk} P^i_{jhr} \right)_{|l|n} = 0 \, ,$$

since y^i is covariant constant . Transvecting (3.20) by y^j and using (1.16b) and (1.15b) ,we get

(3.21)

$$b_{nlm} H^i_{kh} + b_{nlh} H^i_{mk} + b_{nlk} H^i_{hm} + \left(H^r_{hm} P^i_{kr} + H^r_{kh} P^i_{mr} + H^r_{mk} P^i_{hr} \right)_{|l|n} = 0 \, .$$

Thus we may conclude

Theorem 3.1 : In *an* R^h*-trirecurrent space , the identities (3.13) , (3.15) and (3.21) hold and the tensors* $C_{ijr} H^r_{kh} + C_{ihr} H^r_{kj} + C_{ikr} H^r_{jh}$; $C_{ikr} H^r_h - C_{ihr} H^r_k$ *and* $C^j_{kr} H^r_h - C^j_{hr} H^r_k$ *are all h-trirecurrent* .

4. P2-LIKE R^h-TRIRECURRENT SPACE

A P2-like space is characterized by

(4.1) $$P^i_{jkh} = \phi_j C^i_{kh} - \phi^i C_{jkh} \; ,$$

where ϕ_j is a non-zero covariant vector field . We must note that a P2-like space is necessarily a P^*-Finsler space characterized by

(4.2) $$P^i_{kh} = \phi C^i_{kh} ,$$

where $P^i_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r$ and $\phi = \phi_j y^j$.

Let us consider a P2-like R^h-trirecurrent space , then we necessarily have the equation (4.1) , (4.2) and the identity (3.16). From (4.1) and the identity (3.16) ,we have

(4.3)
$$R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + \phi_j \left(H^r_{hm} C^i_{kr} + H^r_{kh} C^i_{mr} + H^r_{mk} C^i_{hr} \right) - \phi^i (H^r_{hm} C_{jkr}$$
$$+ H^r_{kh} C_{jmr} + H^r_{mk} C_{jhr}) = 0,$$

where $R^i_{jhk} y^j = k^i_{jhk} y^j = H^i_{hk}$.

Using (2.18) , (1.10b), $R_{ijhk} = k_{ijhk} + C_{ijm} H^m_{hk}$ and $k_{jikh} + k_{hijk} + k_{kihj} = 0$ in (4.3), we get

(4.4)
$$R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + \phi_j \left(R^i_{khm} + R^i_{mkh} + R^i_{hmk} \right) - \phi^i (R_{jhmk} + R_{jmkh} + R_{jkhm}) = 0,$$

since the metric tensor is *h*-covariant constant the equation (4.4) may be written as

(4.5)
$$R_{ijkh|m} + R_{ijmk|h} + R_{ijhm|k} = \phi_i (R_{jhmk} + R_{jmkh} + R_{jkhm}) + \phi_j (R_{ikhm} + R_{imkh} + R_{ihmk})$$
.

Transvecting (4.5) by y^i , we have

(4.6)
$$H_{.jkh|m} + H_{.jmk|h} + H_{.jhm|k} = \phi (R_{jhmk} + R_{jmkh} + R_{jkhm}) + \phi_j (H_{.khm} + H_{.mkh} + H_{.hmk})$$
,

where $\phi = \phi_i y^i$. We know that the tensor H^i_{kh} satisfies the identity (1.23c) . Differentiating (1.23c) partially with respect to y^j and using (1.19b) and $g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i$,we get

(4.7) $$g_{ij} H^i_{kh} + y_i H^i_{jkh} = o .$$

Taking skew symmetric part of (4.7) with respect to indices *j*, *k* and *h* and using the first Bianchi identity for H^i_{jkh} ,we get

(4.8) $$g_{ij}H^i_{kh}+g_{ih}H^i_{jk}+g_{ik}H^i_{hj}=0 \; .$$

Using $H_{jk.h}\coloneqq g_{ik}H^i_{jh}$ in (4.8) , we may also write

(4.9) $$H_{.jkh}+H_{.hjk}+H_{.khj}=0 \, .$$

Using (4.9) in (4.6), we get

(4.10) $$H_{.jkh|m}+H_{.jmk|h}+H_{.jhm|k}=\phi(R_{jhmk}+R_{jmkh}+R_{jkhm}) \, ,$$

or

(4.11) $$H^i_{kh|m}+H^i_{mk|h}+H^i_{hm|k}=-\phi(R^i_{hmk}+R^i_{mkh}+R^i_{khm}) \; .$$

Differentiating (4.11) covariantly with respect to x^l in the sense of Cartan we get

(4.12)
$$H^i_{kh|m|l}+H^i_{mk|h|l}+H^i_{hm|k|l}=-\phi_{|l}(R^i_{hmk}+R^i_{mkh}+R^i_{khm})-\phi(R^i_{hmk}+R^i_{mkh}+R^i_{khm})_{|l} \; .$$

Differentiating (4.12) covariantly with respect to x^n in the sense of Cartan and using (2.10) ,we get

(4.13)
$$b_{nlm}H^i_{kh}+b_{nlh}H^i_{mk}+b_{nlk}H^i_{hm}=-\phi_{|l|n}(R^i_{hmk}+R^i_{mkh}+R^i_{khm})-\phi_{|l}(R^i_{hmk}+R^i_{mkh}+R^i_{khm})_{|n}$$
$$-\phi_{|n}(R^i_{hmk}+R^i_{mkh}+R^i_{khm})_{|l}-\phi(R^i_{hmk}+R^i_{mkh}+R^i_{khm})_{|l|n} .$$

Thus, we conclude:

Theorem 4.1 : *In a P2-like* R^h *-trirecurrent space we have the following identities :*

$$H^i_{kh|m}+H^i_{mk|h}+H^i_{hm|k}=-\phi(R^i_{hmk}+R^i_{mkh}+R^i_{khm}) \quad ,$$

$$H^i_{kh|m|l}+H^i_{mk|h|l}+H^i_{hm|k|l}=-\phi_{|l}(R^i_{hmk}+R^i_{mkh}+R^i_{khm})-\phi(R^i_{hmk}+R^i_{mkh}+R^i_{khm})_{|l}$$

and

$$b_{nlm}H^i_{kh}+b_{nlh}H^i_{mk}+b_{nlk}H^i_{hm}=-\phi_{|l|n}(R^i_{hmk}+R^i_{mkh}+R^i_{khm})-\phi_{|l}(R^i_{hmk}+R^i_{mkh}+R^i_{khm})_{|n}$$
$$-\phi_{|n}(R^i_{hmk}+R^i_{mkh}+R^i_{khm})_{|l}-\phi(R^i_{hmk}+R^i_{mkh}+R^i_{khm})_{|l|n} .$$

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