On Fuzzy Star Refinement of Open Covering and imensions of Fuzzy Topological Spaces.

AbdulGawad A.Q.Al-Qubati (Ph.D)
Dep. of Mathematics ,University of Hodeidah, Hodeidah, Yemen
E- mail ; gawad202001@yahoo.com

Abstract:

In this paper the concepts of star-refinement and strongly star-refinement of covering are extended to fuzzy topological space in the sense of Chang, basic theorem for covering dimension of normal fuzzy topological space is proved . Also, the small inductive dimension function is extended to fuzzy topological space, and some results for this inductive dimension in Chang'^s space are obtained .

Keywords: Fuzzy topology, star-refinement, covering dimension ,small inductive dimension,

1.Introduction :

The concept of fuzzy set was introduced and studied by Zadeh [16] and the concept of fuzzy topological spaces by Chang [5].Many mathematicians have contributed toed
the development of fuzzy topological spaces. The concept of covering dimension , small inductive dimension and large inductive dimension were studied by many authors [1,2,3,4,6]they used the notion of fuzzy covering of Chang's spaces and Generalized Chang's spaces (GF-Space) for inductive space. In section 3 of this paper we introduce the concept of star- refinement and strongly star- refinement of fuzzy covering, and then by using the concept of covering dimension(see [3,6,12])and it's extension to fuzzy sitting that used in [4] ,we obtained result for fuzzy normal topological space .

In Section 4 the small inductive dimension function for fuzzy topological spaces is introduced and studied, several results are obtained.

2.Preliminaries:

Throughout this paper X will be a non-empty set of points, a fuzzy set A in X is characterized by a membership function μ_A from X to the closed unit interval I = [0,1], $\mu_A(x)$ is denoted to the grade of membership of x in A. The grades 1 and 0 representing respectively full membership and non-membership in a fuzzy set A denoted by 0_X, 1_X respectively.

Clearly an ordinary set is a special case of fuzzy set, any subset of X can be regarded as a fuzzy set in X called crisp fuzzy set. A fuzzy point $p_{x_0}^{\alpha}$ in X is a special fuzzy set in X with membership function defined by: $\mu_{p_{x_0}^{\alpha}}(x) = \begin{cases} \alpha, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}$,

where $0 < \alpha < l$, $p_{x_0}^{\alpha}$ is said to have support x_0, value α, and is denoted by $p_{x_0}^{\alpha}$ or p.

$P_{x_0}^{\alpha} \subset A$ if and only if $\alpha \leq \mu_A(x_0)$, in particular $p_{x_0}^{\alpha} \subset p_{y_0}^{\beta}$ *if* and only if $x_0 = y_0$, $\alpha \leq \beta$. .Other properties of fuzzy sets can be found in [3,16].

Definition 2.1[5]:A family τ of fuzzy sets in X, which satisfy the following conditions:

(1) $0_X, 1_X \in \tau$

(2) If $A, B \in \tau$, then $A \cap B \in \tau$

(3) If $A_\lambda \in \tau$ for each λ in Λ, then $\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$

is called a fuzzy topology for X, and the pair (X, τ) is a fuzzy topological space (or fts for short). We sometimes write **X** or **X** = (X, τ).

Also Chang's fuzzy topological space are generally referred to as Chang's spaces.

Every member of τ is called a τ-open fuzzy set (or simply open fuzzy set) and its complement is a τ-closed fuzzy set (or simply closed fuzzy set).

As an ordinary topology, the indiscrete fuzzy topology contains only $\mathbf{0}_X$ and 1_X (i.e. $\varnothing$ and X), while the discrete fuzzy topology contains all fuzzy sets.

Definition2.2 An open fuzzy set A in fts **X** is said to be clopen if its complement $1_X - A$ is an open.

Definition 2.3[5] Let A be a fuzzy set in fts **X** The closure $\overline{A}$ and interior A° of A are defined, respectively by $\overline{A} = \bigcap \{F : A \subseteq F, F \text{ is closed fuzzy set}\}$, i.e. the intersection of all closed fuzzy sets containing A, and $A^{\circ} = \bigcup \{U: U \subseteq A, U \text{ is open fuzzy set}\}$, i.e. the union of all open fuzzy sets contained in A. If A and B are fuzzy sets in a fts X one can verify that:

(i) A is open (resp. closed) in X if and only if $A = A^{\circ}$ (resp. $A = \overline{A}$)

(ii) $A \subset B$ then $A^{\circ} \subset B^{\circ}$ and $\overline{A} \subset \overline{B}$

(iii) $\overline{A} = 1_X - (1_X - A)^{\circ}$.

The concept of the boundary of fuzzy subset was introduced and studied by Warren [14], as in the following definition.

Definition 2.4. Let A be a fuzzy set in an fts X. The fuzzy boundary of A denoted by $\partial(A)$ is defined as the infimum of all the closed fuzzy sets F in X with the property, $F(x) \geq \overline{A}(x)$ for all $x \in X$, for which $(\overline{A} \cap \overline{A^c})(x) > 0$.

It follows from this definition that ∂A is a closed fuzzy set, Since $\overline{A} \subseteq \overline{A}$ it follows that, $\partial A \subseteq \overline{A}$.

The following results of Warren [14] are needed in the sequel.

Theorem 2.5. Let A and B be fuzzy sets in an fts **X**. Then the following results are hold :

(1) ∂ (A) = 0 if and only if A is open, closed, and crisp.

(2) $\partial(A \cap B) \leq \partial (A) \cup \partial (B)$.

For other undefined elementary concepts and notions in this paper ,we refer to [3,10 ,13].

Definition 2.6 A fts. **X** is said to be normal fuzzy topological space (N-fts. for short) if and only if for every closed fuzzy set F in **X** and every open fuzzy set U in **X** such that $F \subseteq U$, there exists an open fuzzy set V in **X** such that $F \subseteq V \subseteq \overline{V} \subseteq U$.

Definition 2.7[5] A family **U** = $\{U_\lambda : \lambda \in \Lambda\}$ of open fuzzy sets in fuzzy topological space **X** = (X, τ) is a fuzzy covering (or covering for short) of a fuzzy set A if and only if $A \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$.

A sub covering of an open fuzzy covering **U** of A is a subfamily of **U** which is still an open fuzzy covering of A.

According to the definition (2-7) and [3], we give the following definition:

Definition 2.8 The family **U** in Definition (2.6) is a fuzzy covering (or. covering for short) of 1_X if and only if $\bigcup_{\lambda \in \Lambda} U_\lambda = 1_X$, and we say that U is a covering of fts **X**, and a collection $V = \{V_\alpha : \alpha \in \Delta\}$ is said to be refinement of U if $\bigcup_{\alpha \in \Delta} V_\alpha = 1_X$, and each V_α is contained in some members U_λ of **U**.

Definition 2.9[5] A fuzzy set A in fts. **X** = (X, τ) is fuzzy compact (or compact for short) if and only if each covering of A has a finite sub covering, and the fts **X** = (X, τ) is compact if and only if each open covering of 1_X has a finite sub covering.

Two fuzzy sets A and B are said to be overlapping (quasi-coincident) if there exists x in X such that $\mu_A(x) + \mu_B(x) > 1$. In this case A and B are said to be overlap at x.
A and B are non-overlapping (disquisi-coincident) if A and B are not overlapping.

Definition 2.10[4] Let X be a nonempty set. A family **U** = $\{U_\lambda\}_{\lambda \in \Lambda}$ of fuzzy sets in X is said to be overlapping family if there exists x ∈ X such that $\mu_{U_\alpha}(x) + \mu_{U_\beta}(x) > 1$, for all α, β ∈ Λ. A family $\{U_\lambda\}_{\lambda \in \Lambda}$ is non-overlapping if it is not overlapping, that is for every $x \in X$, there exist $\alpha, \beta \in \Lambda$. Such that $\mu_{U_\alpha}(x) + \mu_{U_\beta}(x) \le 1$

The following definition based on the notion of overlapping (quasi- coincident) and it is more suitable for the fuzzy setting.

Definition 2.11[4] Let X be a nonempty set. A family $\mathbf{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of fuzzy sets in X is said to be of order n ($n > -1$) written $ord_f\mathbf{U} = n$, if n is largest integer such that there exists an overlapping subfamily of **U** having $n + 1$ elements.

Remark 2.12 From the above definition if $ord_f\mathbf{U} = n$ then for each $n + 2$ distinct indexes $\lambda_1, \lambda_2, \ldots, \lambda_{n+2} \in \Lambda$,we have $U_{\lambda_1} \cap U_{\lambda_2} \cap \ldots \cap U_{\lambda_{n+2}} = \varnothing$. Then it is non-overlapping, in particular if $ord_f\mathbf{U} = -1$, then **U** consists of the empty fuzzy sets and $ord_f\mathbf{U} = 0$, then U consist of pair wise disjoint fuzzy sets which are not all empty.

Ajmal and Kohli[4] introduced the notion of fuzzy covering dimension by the following definition:

Definition 2.13: The covering dimension of a fts **X** denoted $dim_f(\mathbf{X})$ is the least integer n such that every finite open cover of 1_X has a finite open refinement of order not exceeding n or $+\infty$ if there exists no such integer.

Thus it follows that $\dim_f(\mathbf{X}) = -1$ if and only if $X = \varnothing$ and $\dim_f(\mathbf{X}) \le n$ if every finite open cover of 1_X has a finite open refinement of order $\le n$. We have $\dim_f(\mathbf{X}) = n$ if it is true that $\dim_f(\mathbf{X}) \le n$, but it is false that $\dim_f(\mathbf{X}) \le n - 1$. Finally $\dim_f(\mathbf{X}) = +\infty$ if for every positive integer n it is false that $\dim_f(\mathbf{X}) \le n$.

Remark 2.14:The notion of covering dimension of a fts **X** is a fuzzy topological invariant. Moreover, the covering dimension of a topological space is n if and only if the covering dimension of its characteristic fts is n.

3.Fuzzy star refinement of open Covering.

Now, we give the following definition for star refinement and strongly
star- refinement of open Covering :

Definition3.1 Let **X** be fts. and $\mathbf{V}=\{V_\gamma:\gamma\in\Gamma\}$, $\mathbf{U}=\{U_\lambda: \lambda\in\Lambda\}$be two fuzzy open covering of 1_X we say **V** is an fuzzy star refinement of **U** if the covering st(p_x^σ, V) is a refinement of **U**. where

$st(p_x^\sigma, V) = \bigcup_{\gamma\in\Gamma}\{V_\gamma \in V : p_x^\sigma \in V_\gamma\}$. i.e. $V_\gamma \subset U_\lambda$ for some $\lambda \in \Lambda$ and for each $\gamma\in\Gamma$.

Also we say that **V** is a fuzzy **strongly star – refinement** of **U** if the covering st(V,**V**) is a refinement of **U**. i.e. every open fuzzy set V∈**V** there exist an open fuzzy set U∈**U** such that st(V,**V**) ⊂ U ,where st(V,**V**) denotes the star of the set V with respect to the cover **U**. i.e. st(V,**V**) $=\bigcup_{\gamma\in\Gamma}\{V_\gamma \in V : V\cap V_\gamma \neq \phi\}$,

for some$\lambda\in\Lambda$and for each$\gamma\in\Gamma$.

It is easy to verify the following lemma.

Lemm3.2.Every finite open cover of normal fuzzy topological spaces **X** has a finite open star refinement.

Now, we prove the following proposition.

Proposition 3.3 A normal fuzzy topological space **X** satisfies $\dim_f$(**X**) ≤ n if each finite open covering of 1_X has a fuzzy star finite open refinement which is finite fuzzy open covering of1_X of order ≤ n.

Proof Suppose $\dim_f$(**X**) ≤ n ,for a normal fts.**X** let **U** = {U_i },I =1, 2, ..,k be a fuzzy open covering of 1_X ,and since **X** is normal ,there exists a fuzzy closed cover **F**={$F_1,F_2,\ldots,F_k$ } of 1_X such that $F_i\subset U_i$, then $F_i \subset U_i \subset 1_X$ which implies that $1_X - F_i = F_i^c$.

Let Δ be the set of non-empty subset of {1,2,…,k} . For each δ in Δ let $V_\delta = (\bigcap_{i\in\delta} U_i)\cap(\bigcap_{i\notin\delta} 1_X - F_i)$ where

$\delta=\{i: p_x^\sigma \in U_i\}$,.Now if p_x^σ is fuzzy point in X i.e. $p_x^\sigma \subseteq 1_X$,Then $p_x^\sigma \subseteq V_\delta$.since $p_x^\sigma \subseteq \bigcup V_\delta$ .if and only if $\exists\, \delta \in \Delta, p_x^\sigma \subseteq V_\delta$ .Thus V $=\{V_\delta\}_{\delta\in\Delta}$ is a finite open cover of 1_X, furthermore **V** is star- refinement of **U**.

For if p_x^σ is fuzzy point in 1_X then $p_x^\sigma \subseteq F_j$ for some j, since $1_X = \bigcup_{j\in\Delta} F_j$,and if $p_x^\sigma \subseteq V_\delta$ then $V_\delta \cap F_j \neq \emptyset$,thus j$\in \delta=\{j: p_x^\sigma \in U_j\}$,and since $F_j \subset U_j$ **,X** is normal there is V_δ in **X** such that $p_x^\sigma \in F_j \subseteq V_\sigma \subseteq \overline{V_\sigma} \subseteq U_j$ so that $V_\delta \subset V_j$ .Hence st(p_x^σ, V) $\subset U_j$ .i.e.$\bigcup\{V_\delta \in V : P_X^\sigma \in V_\delta\} \subset U_j$.

Now, since $\dim_f$(**X**) $\leq$ n, there exists a finite open refinement **W** of **V** such that the order of W does not exceed n, since **V** is star- refinement of **U**, it follows that **W** is star- refinement of **U**. ■

4.Small inductive Dimension

In [3] we studied the dimension function of Generalized fuzzy topological spaces (GFS(X, A, τ, φ)) that dimension function are small inductive dimension and large inductive dimension, in particular for small inductive dimension the following theorems and their proofs have been introduced in [1, 2, 3, 6]

Theorem 4.1 If **X** = (X, A, τ, φ) and **Y**= (Y, B, σ, ψ), are two GF-spaces which are homeomorphic to each other, then ind(**X**) = ind(**Y**).

Theorem 4.2 If **X** = (X, A, τ, φ) is a GF-space and **Y** = (Y, B, τ|Y, φ|Y) is a subspace of X induced by fuzzy sets i and s, then ind(**X**) ≥ ind(**Y**).

Theorem 4.3 If **X**= (X, A, τ, φ) is a GF- space such that ind (**X**) < ∞, then for every
n < ind(**X**), there is a subspace **Y** of X such that ind(**Y**) = n.

Theorem 4.4 A non-empty GF-space **X** = (X, A, τ, φ) satisfies ind(**X**) = 0, if and only if for every point p in X and every open set U containing p, there is a clopen set V, such that p ⊆ V ⊆ U.

Theorem 4.5 Let **X** = (X, A, τ, φ) be an FT_3GF-space such that ind(**X**) = 0. Then **X** is totally disconnected.

Here, we introduce the concept of small inductive dimension function for fuzzy topological spaces as Chang'^s space..

Definition 4.6. Let **X** = (X, τ) be a fuzzy topological space. The small inductive dimension of **X**, denoted by indf **X**, is

defined as follows. indf **X** = −1 if X = ∅. For any nonnegative integer n, indf **X** ≤ n if for each x ∈ X and each open fuzzy set G such that G(x) > 0 there exists an open fuzzy set U in X such that U(x) > 0, U ≤ G and indf∂(U) ≤ n −1. indf **X** = n if indf **X** ≤ n is true and indf **X** ≤ n−1 is not true.
indf **X** =∞ if there is no integer n such that indf **X** ≤ n.

Note that if **X** is a general topological space, then this concept reduces to that of ind.
A subset theorem for indf is proved in the following.

Theorem 4.7. If A is a crisp subset of an fts **X**, then indf A ≤ indf **X**.

Proof. The proof is by induction on n. For n = −1, if indf **X** ≤−1, then indf **X** = −1, so that X = ∅. Since A is a crisp subset of X, it follows that A = ∅, and therefore indf A= −1,that is, indf A≤−1. Thus if indf **X** ≤−1, then indfA ≤−1. Therefore the result holds for n = −1.
Assume the result is true for n −1. Then, we shall prove the result for n, that is ,we shall prove that if indf **X** ≤ n, then indf A ≤ n, let indf **X** ≤ n .Then to prove indf A ≤ n, let x∈ A and let G be an open fuzzy set in A, such that G(x) > 0. Since G is open in A by induced fuzzy topology on A , there exists an open fuzzy set H in X such that G = A∩H. Now G(x) > 0 implies H(x) > 0 and A(x) > 0. Since indf **X** ≤ n, H is an open

fuzzy set in X such that H(x) > 0. By definition (4.6) there exists an open fuzzy set V in X such that V(x) > 0,V ≤ H, and indf∂ (V) ≤ n − 1. Let U = A ∩ V. Since V is an open fuzzy set in X, it follows that U is an open fuzzy set in A. Now U(x) > 0. We have A(x) > 0 and V(x) > 0.Therefore A(x)∩V(x) > 0, so that (A∩V)(x) > 0, and hence U(x) > 0. Also U ≤ G. We have V ≤ H. Therefore A∩V ≤ A∩H, so that U ≤ G. Further, indf $\partial_A(U) = \partial_A(A\cap V) \leq \partial_A(A) \cup \partial_A(V) = 0 \cup \partial_A(V) = \partial_A(V) \leq A\cap\partial(V) \leq \partial(V)$.

Thus $\partial_A(U) \leq \partial(V)$. Since indf∂(V) ≤ n −1, by induction hypothesis it follows that indf $\partial_A(U) \leq n - 1$. Thus, for each x ∈ A and each open fuzzy set G in A such that G(x) > 0, there exists an open fuzzy set U in A such that U(x) > 0, U ≤ G, and indf ∂A(U) ≤ n −1. Therefore by the use of definition it follows that indf A ≤ n.

Thus if indf **X** ≤ n, then indf A ≤ n.

Therefore the result holds for n. Hence indf A ≤ indf **X**. ■

Now, for zero dimensionality, we prove the following theorem, for Chang'[s] space.

Theorem 4.8 A non-empty fts **X** = (X,τ) satisfies ind(**X**) = 0 if and only if for every point p in X and every open set U containing p, there is a clopen set V, such that p⊆ V ⊆ U.

Proof Suppose that ind(**X**) = 0, then by the definition ind(X) ≤ 0 if for every point p in X and every open set U containing p there exists an open set V such that $p \subseteq V \subseteq U$, and $ind(\partial V) \leq 0-1= -1$, and since $ind(\mathbf{X}) \leq 0-1$, then there is a point p in X and open set U, $p \subseteq U$, and for every open set V such that $p \subseteq V \subseteq U$ and $ind(\partial V) \leq 0-2$. Thus one such open set V must satisfy the condition, $ind(\partial V) = -1$ implies that $\partial V = \emptyset$. Therefore the only set with empty boundaries are clopen. Then V is clopen set.

Conversely for every point p in X and every open set U containing p, there exists
a clopen set V such that $p \subseteq V \subseteq U$ and $ind(\partial V) = ind\emptyset) = -1$.Then $ind(\mathbf{X}) = 0$. ■

The following corollary is obvious.

Corollary 4.9 Every non-empty subspace of zero dimensional fts **X** is zero dimensional.

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