ANALYTIC SOLUTIONS FOR EKMAN BOUNDARY LAYER ON A POROUS PLATE

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ABSTRACT:

A model of Ekman boundary layer on a porous Plate is considered and the governing differential equations are introduced. Steady state solutions for the velocity and temperature distributions are obtained. The effect of buoyancy forces on velocity and temperature distributions are studied. The results indicate that the effects of buoyancy forces have significant contribution to the field of profiles.

1. Introduction

Ekman boundary layer on arbitrary surface which forms in a rotating systems at either free or solid boundaries, typically normal to the axis of rotation, such boundary layers play an important roles in many geophysical and technical flows including large atmospheric vortices and source-sink flow in turbines[5]. Further, when a vast expansion of viscous liquid bounded by an infinite flat plate is rotating about an normal axis to the plate, a layer is formed near the plate where the viscous and coriolis are of the same order of magnitude, this is known as Ekman layer [5].

The study of the Ekman layer on arbitrary surface is of geophysical interest as it has seen in the study of wind-generated ocean currents on a rotating earth [3]. The dynamic of fluids in porous media has been a subjects of numerous theoretical and experimental studies because of its importance for engineering and environmental application [10]. The exact analytic solution for free convection boundary layers on a heated vertical plate with lateral mass flux embedded in a saturated porous medium have been obtained by Magyn and Keller[9]. Mosa and Mnaa [12] investigated the effect of radiation limits in the MHD Ekman layer on a porous plate, showing that there was a significant effect of radiation on the temperature distribution. Wang and Hayat [14] developed the governing equations for the unsteady hydromagnetic rotating flow of a fourth order fluid past a porous plate. A numerical solution to the problem of the three-dimensional fluid flow in a long rotating heterogeneous porous channel is obtained by Havstad and Vadasz [7].

Al makrami [1,2] has found the steady state solution for Ekman boundary layer flow and the stability of this model under an optically thick limit and under an optically thin limit has been investigated.

In this paper, the model of Ekman layer on porous plate has been considered. The equations governing such a model have been introduced and the effect of buoyancy forces on the velocities and temperatures profile are studied.

2. Formulation

The system consists of fluid occupying the half-space below an infinite horizontal plate which taken in the xy-plane. The entire fluid plane system is rotates a bout a vertical downward axis, which is taken as a positive direction of z-axis, with a uniform angular velocity $\vec{\Omega} = \Omega(0,0,-1)$, the infinite horizontal plate which coincides with z=0 plane, is uniformly porous, and its temperature is considered having a constant value equal to T_0 . The free stream temperature is T_1 .

3.Governing equations

The basic differential equations governing such a model are state equation, continuity equation, motion equation and energy equation. They can be summarized as follows [12]

$$\rho = \rho_0 [1 - \beta (T - T_0)],$$

(3.1)

$$\nabla . \vec{V} = 0 \quad ,$$

(3.2)

$$\rho_0 \left\{ \frac{\partial \vec{V}}{\partial t} + \vec{V} . \nabla \vec{V} + 2 \vec{\Omega} \times \vec{V} \right\} = -\nabla p + \mu \nabla^2 \vec{V} + \rho \vec{g} - \frac{\mu}{K} \vec{V}$$

and

$$\rho_0 c_p \left\{ \frac{\partial T}{\partial t} + \vec{V} . \nabla T \right\} = k_1 \nabla^2 T + \Phi$$

(3.4)

Where ρ is density, β is the coefficient of thermal expansion, and ρ_0 is a characteristic density. This means that the density is constant everywhere except where it produce buoyancy force $\vec{V} = U \vec{i} + V \vec{j} + W \vec{k}$ is the velocity vector where, U,V and W are the velocity components in x,y, and z directions respectively, μ is the fluid viscosity, P is the pressure, g is the acceleration gravitaty, K is the permeability coefficient of the porous medium, c_p is specific heat at constant pressure, T is the temperature, k_1 is the coefficient of thermal conductivity and Φ is the viscouse dissipation. Equation(3.1) is known as Boussinesq approximation. Equation (3.2) becomes

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

(3.5)

This means that U and V are functions of z only [5].

The fourth term on the right hand side of equation (3.3) represents the effect of permeability of porous medium .

The motion equation(3.3) is reduced to

$$-2\Omega\rho_0 V = -\frac{\partial P}{\partial x} + \mu\frac{d^2U}{dz^2} - \frac{\mu}{K}U \ , \tag{3.6}$$

$$2\Omega\rho_0 U = -\frac{\partial P}{\partial y} + \mu\frac{d^2V}{dz^2} - \frac{\mu}{K}V \tag{3.7}$$

and

$$0 = \frac{\partial P}{\partial z} - \rho g \tag{3.8}$$

For such a model, the energy equation is

$$\rho_0 c_p\left[U\,\frac{\partial T}{\partial x}\right] = k_1\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) + \left[\left(\frac{dV}{dz}\right)^2 + \left(\frac{dU}{dz}\right)^2\right] \tag{3.9}$$

The boundary conditions satisfying the governing equations are

I) For velocity:

$$U = V = W = 0 \qquad when \quad z \to 0$$
$$U = u_0\,, V = W = 0 \quad when \quad z \to \infty \tag{3.10}$$

where u0 is constant, reference velocity

II) For temperature:

$$T = T_0 \quad when \quad z \to 0$$
$$T = T_1 \quad when \quad z \to \infty$$

(3.11)

The equation of motion (3.3) without fourth term on the right hand side, subjected to the boundary conditions (3.10),have been solved by Almakrami [1]. Eliminating pressure term from equation (3.5), (3.6) and (3.7) it follows that

$$2\Omega\frac{dv}{dz} = -\frac{1}{\rho_0}\frac{\partial^2 P}{\partial x \partial z} + \frac{\mu}{\rho_0}\frac{d^3 u}{dz^3} - \frac{\mu}{kp_0}\frac{du}{dz}$$

(3.12)

$$2\Omega\frac{du}{dz} = -\frac{1}{\rho_0}\frac{\partial^2 P}{\partial y \partial z} + \frac{\mu}{\rho_0}\frac{d^3 v}{dz^3} - \frac{\mu}{k\rho_0}\frac{dv}{dz}$$

(3.13)

$$0 = -\frac{1}{\rho_0}\frac{\partial^2 P}{\partial z \partial x} - \frac{\partial p}{\partial x}g$$

(3.14)

$$0 = -\frac{1}{\rho_0}\frac{\partial^2 P}{\partial z \partial y} - \frac{\partial p}{\partial y}g$$

(3.15)

Using the state Equation(3.1), and letting P=P(x,y) [5] , then we get

$$\frac{\partial T}{\partial x} = \frac{T_0}{\rho_0 g}\left(2\Omega\frac{dV}{dz} + \frac{\mu}{\rho_0}\frac{d^3 U}{dz^3} - \frac{\mu}{K\rho_0}\frac{dU}{dz}\right)$$

(3.16)

$$2\Omega\frac{dU}{dz}=\frac{\mu}{\rho_0}\frac{d^3V}{dz^3}-\frac{\mu}{K\rho_0}\frac{dV}{dz}$$

(3.17)

Since the right hand side of equation (3.16) is a function of z alone, then

$$T=T^*(z)+\frac{u_0}{\nu}\tau x\overline{T}(z)$$

(3.18)

where τ is a suitable scalar. This is means that T is a linear function of *x* .

Substituting (3.18) in the energy equation (3.9) we get

$$\rho_0 c_p\left\{U\frac{u_0}{\nu}\tau\overline{T}\right\}=k_1\left(\frac{\partial^2T^*}{\partial z^2}+\frac{u_0}{\nu}\tau x\frac{\partial^2\overline{T}}{\partial z^2}\right)+\left(\frac{dV}{dz}\right)^2+\left(\frac{dU}{dz}\right)^2$$

(3.19)

Equation (3.19) can be rewritten as

$$\rho_0 c_p\frac{\tau u_0}{\nu}U\overline{T}=k_1\frac{\partial^2T^*}{\partial z^2}+\left(\frac{dV}{dz}\right)^2+\left(\frac{dU}{dz}\right)^2$$

(3.20)

$$k_1\frac{\tau u_0}{\nu}\frac{d^2\overline{T}}{dz^2}=0$$

(3.21)

After linearizing T the initial and boundary conditions for temperature become

$$T^* = T_0 \quad when \quad z = 0$$
$$T^* = T_1 \quad when \quad z = \infty$$
$$\overline{T} = 0 \quad when \quad z = 0$$
$$\overline{T} = 0 \quad when \quad z = \infty$$

(3.22)

4. Dimensionless form of the governing equations

For converting the governing partial differential equations (3.16), (3.17), (3.18), (3.20) and (3.21) into dimensionless, we introduce the following dimensionless quantities as follows

$$x=\frac{\nu}{u_0}\xi, y=\frac{\nu}{u_0}\eta,\ z=\frac{\nu}{u_0}\zeta, T=T_0\theta,\ U=u_0u,\ V=u_0v$$

where ξ,η *and* ζ are the dimensionless distance in ξ , η and ζ directions respectively,θ is the dimensionless temperature and *u* and *v* are dimensionless velocities in ξ and η directions respectively.

Define:

$\mathrm{Re} = \frac{\nu\rho_0}{\mu}$ Reynolds number, $\mathrm{E} = \frac{2\Omega\nu}{u_0^2}$ Rotation number,

$\mathrm{P}r = \frac{c_P\mu}{k_1}$ Prandtl number $\mathrm{E}c = \frac{u_0}{c_P T_0}$ Eckert number, $Gr = \frac{g\rho_0\beta\nu^2}{u_0{}^3\mu}$ Grashof number .

Substituting these dimensionless quantities into equations (3.16), (3.17), (3.18), (3.20) and (3.21),we obtain

$$\frac{d\theta}{d\xi}=\frac{E\,\mathrm{Re}}{Gr}\frac{dv}{d\zeta}+\frac{1}{Gr}\frac{d^3u}{d\zeta^3}-\frac{1}{kGr}\frac{du}{d\zeta}$$

(4.1)

$$\frac{du}{d\zeta}=\frac{1}{E\,\mathrm{Re}}\frac{d^3v}{d\zeta^3}-\frac{1}{kE\,\mathrm{Re}}\frac{dv}{d\zeta}$$

(4.2)

$$\theta=\theta^*(\zeta)+\tau\varsigma\,\overline{\theta}\,(\zeta)$$

(4.3)

$$\tau u\,\overline{\theta}=\frac{1}{pr}\frac{d^2\theta^*}{d\zeta^2}+\frac{Ec}{\mathrm{Re}}\left[\left(\frac{dv}{d\zeta}\right)^2+\left(\frac{du}{d\zeta}\right)^2\right]$$

(4.4)

$$\frac{1}{pr}\frac{d^2\overline{\theta}}{d\zeta^2}=0$$

(4.5)

The initial and boundary conditions (3.10) and (3.22) become

I) For velocity

$$\begin{aligned}
&u=0 \quad && when\ \ \zeta\to 0\\
&u=1 \quad && when\ \ \zeta\to\infty\\
&v=0 \quad && when\ \ \zeta\to 0\\
&v=0 \quad && when\ \ \zeta\to\infty
\end{aligned}$$

(4.6)

II) For temperature

$\overset{*}{\theta} = 1 \quad when \quad \zeta \to 0$

$\overset{*}{\theta} = \Theta \quad when \quad \zeta \to \infty$

$\bar{\theta} = 0 \quad when \quad \zeta \to 0$

$\bar{\theta} = 0 \quad when \quad \zeta \to \infty$

(4.7)

where $\Theta = \dfrac{T_1}{T_0}$

5. Solution

Equation (4.5) give

$$\bar{\theta} = c_1 \zeta$$

(5.1)

Where c_1 is arbitrary constant. Then Equation (4.1) becomes

$$\tau c_1 \zeta = \frac{E\,\mathrm{Re}}{Gr}\frac{dv}{d\zeta} + \frac{1}{Gr}\frac{d^3u}{d\zeta^3} - \frac{1}{kGr}\frac{du}{d\zeta}$$

(5.2)

Equations (4.2) and (5.2) may be reduced to one third–ordinary linear differential equation of complex variable $w=u+iv$ where $i = \sqrt{-1}$ as:

$$\frac{d^3w}{d\zeta^3} - iE\,\mathrm{Re}\frac{dw}{d\zeta} - \frac{1}{k}\frac{dw}{d\zeta} = \tau c_1 Gr \zeta$$

or

$$\frac{d^3w}{d\zeta^3} - d^2\frac{dw}{d\zeta} = \tau c_1 Gr \zeta$$

(5.3)

where $d^2 = (E\,\mathrm{Re}\,i + \frac{1}{k})$

With the boundary conditions

$$w(0) = 0$$
$$w(L) = 1 \quad , L \to \infty (L >> 1)$$
$$\frac{d^3 w}{d\zeta^3} = \left(\tfrac{1}{k} + E\,\mathrm{Re}\,i\right)\frac{dw}{d\zeta}(0)$$

(5.4)

Solving this equation, using the relevant boundary conditions, by considering $c_1 = \frac{1}{L^2}$ and neglecting smallest quantities, we obtain the following

$$w = (1 + A_2 - A_1 i) - (1 + A_2 - A_1 i)e^{-d\zeta} + A\zeta^2$$

(5.5)

where

$$A = \frac{-\tau\, Gr\, c_1}{2(E\,\mathrm{Re}\,i + \frac{1}{k})} = c_1 A_1 i - c_1 A_2 \ , A_1 = \frac{E\,\mathrm{Re}\,Gr\,\tau}{2(E^2\,\mathrm{Re}^2 + \frac{1}{k^2})} , A_2 = \frac{Gr\,\tau}{2k(E^2\,\mathrm{Re}^2 + \frac{1}{k^2})} \ ,$$

$$d = ci + b \ , \ c = \{\frac{1}{2k} + \frac{1}{2}(\frac{1}{k^2} + E^2\,\mathrm{Re}^2)^{\frac{1}{2}}\}^{\frac{1}{2}} \ , b = \frac{E\,\mathrm{Re}}{2c}$$

since $w = u + iv$ then

$$u = (1 + A_2)(1 - \cos(b\zeta)e^{-c\zeta}) + A_1 \sin(b\zeta)e^{-c\zeta} - c_1 A_2 \zeta^2$$

(5.6)

and

$$v = A_1(\cos(b\zeta)e^{-c\zeta} - 1) + (1 + A_2)\sin(b\zeta)e^{-c\zeta} + c_1 A_1 \zeta^2$$

(5.7)

Substitute from (5.6), (5.7), and (5.1) into (4.4) we get

$$\frac{d^2\theta^*}{d\zeta^2} = R_1\zeta[1-\cos(b\zeta)e^{-c\zeta}] + R_2\zeta\sin(b\zeta)e^{-c\zeta} + R_3\zeta^3 + R_4 e^{-2c\zeta} + R_5\zeta^2$$
$$+ R_6\zeta\, e^{-c\zeta}\cos(b\zeta)$$

(5.8)

where

$$R_1 = \Pr\pi c_1(1+A_2), R_2 = -\frac{4c_1^2 \Pr Ec A_2}{\mathrm{Re}}(b + bA_1 + bA_1 - c + cA_1 - cA_2) + \Pr\pi c_1 \ ,$$
$$R_3 = -\Pr\pi c_1{}^2 A_2 \ , R_4 = -\frac{\Pr Ec}{\mathrm{Re}}(1 + 2A_2 + A_2^2 + A_1^2)(c^2 + b^2) \ , R_5 = -\frac{4c_1 \Pr Ec}{\mathrm{Re}}(A_2^2 + A_1^2)$$
$$R_6 = -\frac{4c_1 \Pr Ec A_2}{\mathrm{Re}}(b - bA_1 + bA_2 + c + cA_1 + cA_2) \ ,$$

Equation(5.8) gives

$$\overset{*}{\theta} = \frac{R_3}{20}\zeta^5 + \frac{R_5}{12}\zeta^4 + \frac{R_1}{6}\zeta^3 + R_{13}e^{-c\zeta}\cos(b\zeta) + R_{14}e^{-c\zeta}\sin(b\zeta) + R_{15}\zeta\, e^{-c\zeta}\cos(b\zeta)$$
$$+ R_{16}\zeta\, e^{-c\zeta}\sin(b\zeta) + \frac{R_4}{4c^2}e^{-2c\zeta} + r_1\zeta + r_2$$

(5.9)

where

$$R_{13} = -\frac{R_7(c^2 - b^2) + 2cbR_8}{(c^2+b^2)^2} - \frac{cR_9 + bR_{10}}{c^2+b^2}, R_{14} = \frac{2cbR_7 - (c^2-b^2)R_8}{(c^2+b^2)^2} + \frac{bR_9 - cR_{10}}{c^2+b^2},$$
$$R_{16} = -\frac{cR_7 + bR_8}{c^2+b^2}, R_{15} = \frac{bR_7 - cR_8}{c^2+b^2}, r_2 = 1 - R_{13} \ , r_1 = \frac{\Theta}{L} - (\frac{R_3}{20}L^4 + \frac{R_5}{12}L^3 + \frac{R_1}{6}L^2) - \frac{R_1}{L}$$
$$R_7 = \frac{c(R_1 - R_6) - bR_2}{c^2+b^2}, R_8 = \frac{b(R_6 - R_1) - cR_2}{c^2+b^2} \ , R_9 = \frac{(c^2-b^2)(R_1 - R_6) - 2cbR_2}{(c^2+b^2)^2}, and$$
$$R_{10} = \frac{2cb(R_6 - R_1) - (c^2-b^2)R_2}{(c^2+b^2)^2}$$

Substituting (5.9) and (5.1) into (4.3) we get the temperature distribution as

$$\theta = \frac{R_3}{20}\zeta^5 + \frac{R_5}{12}\zeta^4 + \frac{R_1}{6}\zeta^3 + R_{13}e^{-c\zeta}\cos(b\zeta) + R_{14}e^{-c\zeta}\sin(b\zeta) + R_{15}\zeta\, e^{-c\zeta}\cos(b\zeta)$$
$$+ R_{16}\zeta\, e^{-c\zeta}\sin(b\zeta) + \frac{R_4}{4c^2}e^{-2c\zeta} + (r_1 + c_1\xi)\zeta + r_2$$
(5.10)

6. Results and conclusions

6.1. Results

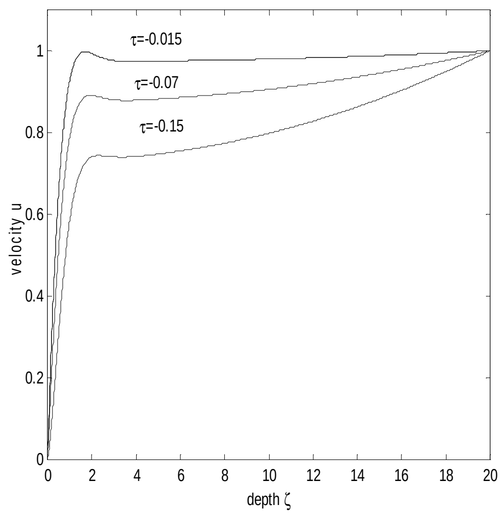

Fig.1. profile of dimensionless velocity *u* for different value of τ

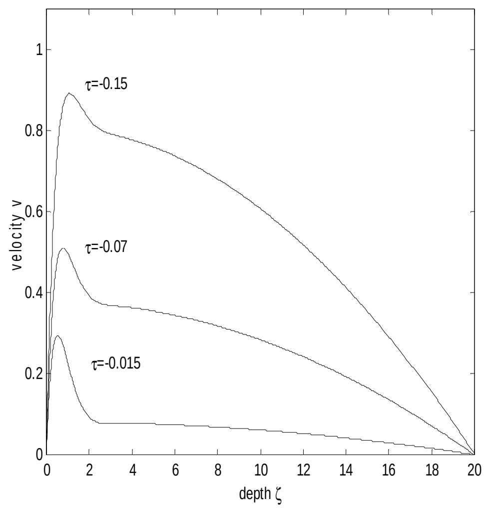

Fig.2. profile of dimensionless velocity *v* for different value of τ

Fig.1. Shows the variation of u with depth ζ at different values of τ, for arbitrary values for the parameters E=0.25,Gr=60, k=0.602 Re=20 and Pr=1.7. It is shown that u start to increase rapidly from the surface(ζ=0) to reach a maximum values u=0.99 at depth approximately 1.7 for τ=-0.015, u=0.89 at depth approximately 1.9

for τ=-0.07,and u=0.74 at depth approximately 2.2 for τ=-0.15, then u increases slowly to reach same value v=1 at
depth 20 for τ=-0.015, τ=-0.7and τ=-0.15.

Fig.(2), shows the variation of v with depth ζ,with parameters E=0.25 ,Gr=60, k=0.602,Re=20 and Pr=1.7. v, start to increase rapidly from the surface to reach a maximum value v_{max} =0.89 at depth a bout 1.08 for τ=-0.15 , a bout v_{max}= 0.51 at depth a bout 0.8 for τ=-0.07 and a bout v_{max} =0.2

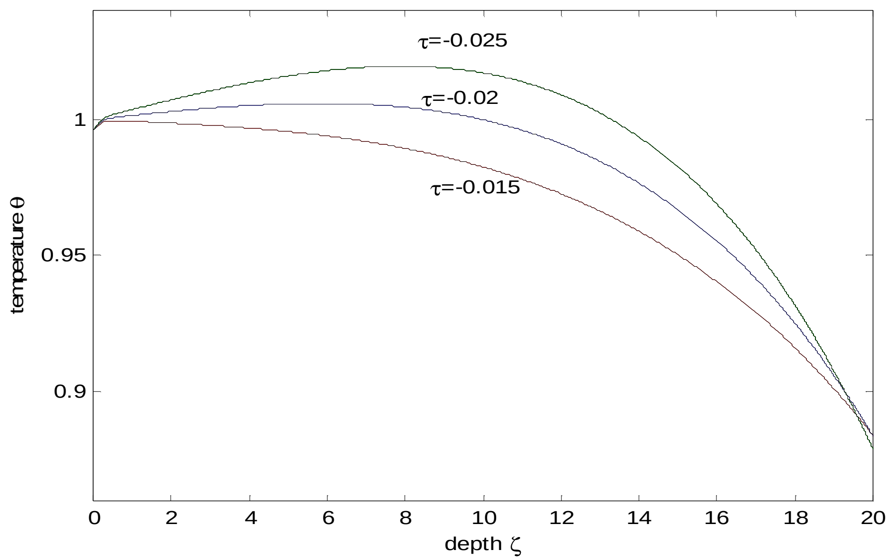

Fig.3. Profile of dimensionless temperature for different value of τ

97 at depth a bout 0.55 for τ=-0.015,then v decreases slowly to reach same value v=0 at depth 20 for τ=-0.15, τ=-0.07andτ=-0.015

Fig.(3),shows the variation of θ with depth ζ,with parameters E=0.25 Gr=60 , k=0.602 , Re=20, Θ=0.99992 , Ec=0.010,ξ=10 and Pr = 1.7 .The temperature increases very slowly with depth to reach a maximum value θ_{max}=1.02 at ζ=8. Then the temperature starting to decrease with increasing depth.

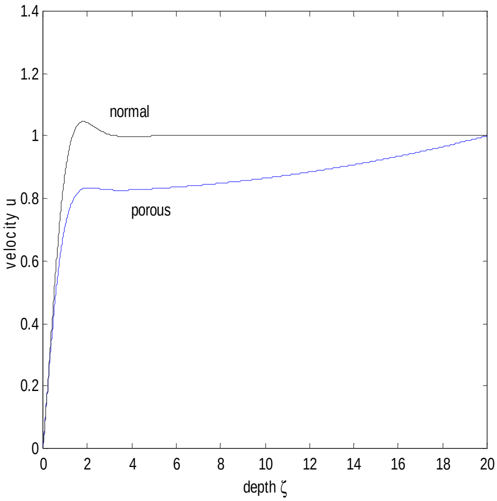

Fig.4. . A comparison between profiles of dimensionless velocities *u*

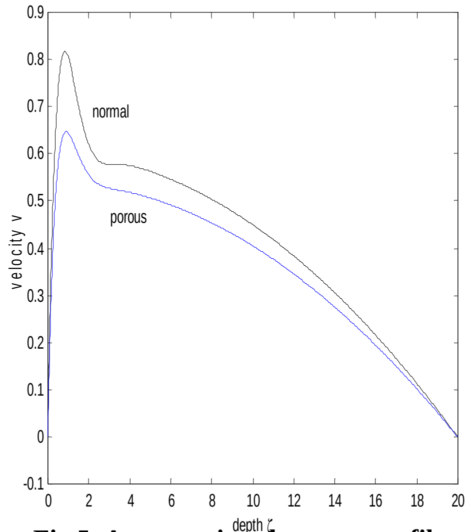

Fig.5. A comparison between profiles of dimensionless velocities *v*

Fig.4. Shows a comparison between profiles of dimensionless velocity *u* with parameters, E=0.25,Gr=60 ,Re=20, τ=-0.1, k=0.602 , Pr=1.7.It is clear that in the porous case, the values of u are smaller than their values in normal case (i.e., when k=0).

Fig.5. Shows a comparison between profiles of dimensionless velocity v with parameters, E=0.25, Gr=60, Re=20, τ=-0.1, k=0.602, Pr=1.7. It is clear that in the porous case, the values of v are smaller than their values in normal case (i.e., when k=0).

6.2. Conclusion

In this paper we have attempted to investigate the effect of the buoyancy forces on an Ekman Layer on a porous flat plate. The systems of differential equations with the relevant boundary conditions have been written . The resulting differential equations represent this model have been solved for velocities given by equations (5.6) & (5.7), and temperature given by equation (5.10) . The results are illustrated by graphs. The appropriate boundary-Layer equation, for this model involve one important parameter τ, which measures the velocities and temperature sensitivities to buoyancy forces arises from variation of density . Effects on velocities and temperature are illustrated in Fig.1.througlh Fig.5. The velocity distributions u for different values of τ are given in Fig.1. It is shown that the values of velocities increases with increasing the values of τ also the velocity distributions v for different values of τ are illustrated in Fig.2., the values of velocities increase with decreasing the values of τ . The temperature distributions for different values of τ are illustrated in figure[3],the values of temperatures increases with decreasing the values of τ. Fig.4. through Fig.5. Show the effect of porosity on the velocities distributions u and v, it has been seen that when their values increasing the decreasing in the velocities profiles appear .

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