NEW TYPES OF COMPACT SPACES

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Abstract :

In this work, we introduce a new types for compact space, namely "s-compact, p-compact, α-compact and β-compact" spaces, all of these are stronger than compact spaces. We discussed and proved several theorems concerning these new concepts.

1. Introduction

Let (X,*T*) be a topological space, and let A be a subset of X. We denote the closure of A by $\overline{A}$, the interior of A by A^o, and a neighborhood of a point p will be denoted by *nbd*. A subset S of (X,*T*) is called

1. Semi-open if $S \subseteq \overline{S^{o}}$,

2. Pre-open (briefly p-open) if $S \subseteq \overline{S}^{\,o}$,

3. α-open if $S \subseteq \overline{S^{o}}^{\,o}$,

4. β-open if $S \subseteq \overline{\overline{S}^{\,o}}$.

This notions were introduced by Mashuour[2], and Levin[10].

A subset W of (X, *T*) is called,

1. Semi-closed if W^c is semi-open,

2. P-closed if W^c is p-open,

3. α-closed if W^c is α-open,

4. β-closed if W^c is β-open.

A space (X, *T*) is said to be semi–compact (P-compact, α-compact, β-compact) space, if every cover of X, by semi–open (p-open, α-open, β-open) sets, has a finite sub cover. If $F=\{w_a : a\in \Lambda\}$, is a cover of X by semi-open (p-open, α-open, β-open) sets in X, then we say that F is semi-open (p-open, α-open, β-open) cover of X. A subset B, of a space X, is Semi-compact (P-compact, α-compact, β-compact), iff every semi–open (p-open, α-open, β-open) cover of A, has a finite sub cover.

Of course, we have that :"

$$
\begin{array}{ccccc}
\text{open set} & \Rightarrow & \alpha-\text{open} & \Rightarrow & s-\text{open} \\
 & & \Downarrow & & \Downarrow \\
 & & p-\text{open} & \Rightarrow & \beta-\text{open}
\end{array}
$$

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While in the opposite direction may not hold, for example,

1. An interval in the form (a, b], is s-open, but not α-open and not open.
2. Let X={1,2,3}, and $T_X=\{\phi,\{1\},X\}$, then the set {2,3} is β-open,

but not s-open, also it is p-open but not α-open, and not open.

It follows that

$$\begin{array}{ccccc} \beta-\text{compact} & \Rightarrow & p-\text{compact} & & \\ \Downarrow & & \Downarrow & & \\ s-\text{compact} & \Rightarrow & \alpha-\text{compact} & \Rightarrow & \textit{compact} \end{array}$$

While in the opposite direction may not hold, for example, if $X=\{x\}\cup\{x_\alpha,\ \alpha\in A$, and A is any uncountable index set$\}$, and $T_X=\{\phi,\{x\},X\}$, then X is compact but not α-compact, and it is easy to find examples for other directions. There are few concepts related to compact space, which are not discussed in this paper, they will be published in a future work. In this work, the word "space" means topological space, and for the set of isolated points of any space X, we use the symbol I_X.

2. First main results

In this section, we proved several theorems about semi-compact (p-compact, α-compact, β-compact) spaces. Firstly, we need the following definitions, which is immediate consequence of the above statements.

Definition 2.1.

Let $q\in X$, we say that q is semi-interior (p-interior, α-interior, β-interior) point of subset A, iff there is semi-nbd (p-nbd, α-nbd, β-nbd) of q, which contained in A,

i. e., $q\in X$, is semi-interior (p-interior, α-interior, β-interior) point of set A, iff there is semi-open (p-open, α-open, β-open) set U in X, such that $q\in U\subset A$, and hence we can say that a subset of a space X, is semi-open (p-open, α-open, β-open) set, iff every it's points is semi-open (p-open, α-open, β-open).

Definition 2.2.

Let $f: X\rightarrow Y$, be a function, we say that f is semi-continuous (p-continuous, α-continuous, β-continuous), iff the inverse image of every semi-open (p-open, α-open, β-open) set in Y, is semi-open (p-open, α-open, β-open) set in X.

Theorem 2.3.

Let f :X→Y is semi-continuous (p-continuous, α-continuous, β-continuous) onto function, if X is semi-compact (p-compact, α-compact, β-compact), then also Y is.

Proof

Let $F=\{w_a : a\in \Lambda\}$, be semi-open (p-open, α-open, β-open) cover of Y, consider $F_i= f^{-1}(w_a)$, $a\in \Lambda$, then F_i is a semi-open (p-open, α-open, β-open) cover of X,

but X is semi-compact (p-compact, α-compact, β-compact), then there exists $\alpha_1,\ldots,\alpha_n$, such that $X\subseteq \bigcup_{i=1}^{n} f^{-1}(w_{ai})$. Now $Y= f(X) \subseteq f(\cup f^{-1}(w_{ai}))$, then $Y \subseteq f(\cup f^{-1}(w_{ai}))$, that means $Y \subseteq \cup w_{ai}$, then Y is semi-compact (p-compact, α-compact, β-compact).

Theorem 2.4.

Let X is semi-compact (p-compact, α-compact, β-compact) space, and A is a semi-closed (p-closed, α-closed, β-closed) subset of X, then A is also semi-compact (p-compact, α-compact, β-compact).

Proof

Let $F=\{w_a : a\in \Lambda\}$ is semi-open (p-open, α-open, β-open) cover of A by semi-open (p-open, α-open, β-open) subset of X, consider $F_i = F \cup \{A^C\}$, then F_l is semi-open (p-open, α-open, β-open) cover of X, since X is semi-compact (p-compact, α-compact, β-compact), then there exists $\alpha_1,\ldots,\alpha_n$, such that $X\subseteq\{\bigcup_{i=1}^{n}(w_{ai}) \cup A^C\}$. So that $A\subseteq \cup (w_{ai})$, hence A is semi-compact (p-compact, α-compact, β-compact). Before we state our next theorem, we need the following definition.

Definition 2.5.

Let (X, *T*) be a topology space, we say that X is semi–T_2space "semi-Hausdorff -space", (p-T_2-space, α-T_2-space, β-T_2-space), iff given a, $b \in X$, $a \neq b$, then there is two disjoint semi-open (p-open, α-open, β-open) subsets U and V of X, Such that $a \in U$, $b \in V$.

Theorem 2.6.

Every semi-compact (p-compact, α-compact, β-compact) subset of semi–T_2-space (p-T_2-space, α-T_2-space, β-T_2-space), is semi-closed (p-closed, α-closed, β-closed).

Proof

Let X be a semi–T_2-space (p-T_2-space, α-T_2-space, β-T_2-space). If $A \subseteq X$ is semi-compact (p-compact, α-compact, β-compact) in X, and $a \in X-A$, $b \in A$, $a \neq b$. Then there is two disjoint semi-open (p-open, α-open, β-open) subsets U_b , V_a of X, such that $b \in U_a$, $a \in V_b$, and $U_b \cap V_a = \phi$. Let $F=\{w_b : b \in \Lambda\}$, then F is semi-open (p-open, α-open, β-open) cover of A. Since A is semi-compact (p-compact, α-compact, β-compact) subset of X, then there exist $b_1, b_2 \ldots, b_n$, such that $A \subseteq \bigcup_{i=1}^{n} w_{bi} = W$. Let $V = \bigcap_{i=1}^{n} V_{ai}$, then V is a semi-open (p-open, α-open, β-open) subset of X containing a, clearly that V does not intersect W, so that $V \subseteq X-A$, and hence a, is an interior point of X-A, so X-A is semi-open (p-open, α-open, β-open) subset in X. Therefore A is semi-closed (p-closed, α-closed, β-closed) subset in X. Also for our next theorem, we need the following definition.

Definition 2.7.

Let $f : X \rightarrow Y$ be a function, we say that f is semi-closed (p-closed, α-closed, β-closed), iff the image of every semi-closed (p-closed, α-closed, β-closed) subset of X, is semi-closed (p-closed, α-closed, β-closed) in Y.

Theorem 2.8.

A semi-continuous (p-continuous, α-continuous, β-continuous) function of semi-compact (p-compact, α-compact, β-compact) space into semi-T_2-space (p-T_2-space, α-T_2-space, β-T_2-space), is semi-closed (p-closed, α-closed, β-closed).

Proof

Let $f : X \rightarrow Y$ is semi-continuous (p-continuous, α-continuous, β-continuous) function, from semi-compact (p-compact, α-compact, β-compact) space X, into semi–T_2-space (p-T_2-space, α-T_2-space, β-T_2-space). Let A is semi-closed (p-closed, α-closed, β-closed) subset of X, then A is semi-compact (p-compact, α-compact, β-compact) (by theorem 2.4), then f (A) is also semi-compact (p-compact, α-compact, β-compact), but f (A) is a subset of a semi–T_2-space (p-T_2-space, α-T_2-space, β-T_2-space) Y, then (by Theorem 2.6), f (A) is semi-closed (p-closed, α-closed, β-closed), so that f is semi-closed (p-closed, α-closed, β-closed) function.

Theorem 2.9.

Let $f : X \rightarrow Y$ be a one-one semi-continuous (p-continuous, α-continuous, β-continuous) function, from semi-compact (p-compact, α-compact, β-compact) space, into semi–T_2-space (p-T_2-space, α-T_2-space, β-T_2-space), then f^{-1} exist and hence will to be semi-continuous (p-continuous, α-continuous, β-continuous).

Proof.

Let F, be any semi-closed (p-closed, α-closed, β-closed) subset of X. It follows that F, is semi-compact (p-compact, α-compact, β-compact), by the continuity of *f*, we have that *f* [F] is semi-compact (p-compact, α-compact, β-compact) subset, of semi–T_2-space (p-T_2-space, α-T_2-space, β-T_2-space). Then *f* [F] is semi-closed (p-closed, α-closed, β-closed) in Y, by hypothesis ($(f^{-1})^{-1}[F]= f[F]$, So that f^{-1} is semi-continuous (p-continuous, α-continuous, β-continuous).

3. Second main results

In this section, we state and prove theorems concerning a new our concepts, whose it's holds for the condition of non empty interior of every infinite subset of any topological space X. It is easy to notice that (R, T_u) has non empty interior for any infinite subset of R, but every finite space and every discrete space does not satisfies this property. The following propositions are clear.

Proposition 3.1.

In a space X, the following statements are equivalent:

1) For any infinite subset of X, has non empty interior.
2) For any $A \subseteq X$, if $A^o = \phi$ then A is finite.
3) For any $A \subseteq X$, $\overline{A} \setminus A$ is finite.

Proposition 3.2.

Let X be any topological space, then $X \setminus I_x$ is finite, iff for any infinite subset of X, has non empty interior.

For p-compact, we have the following theorem.

Theorem 3.3.

Any a space X is p-compact, iff it is compact and for any infinite subset of X, has non empty interior.

Proof

In the first, a p-compact space is compact, and for the second part, let $A \subseteq X$ such that $A^o = \phi$, i. e. X-A is dense, hence for each $x \in A$, if $S_{\mathbf{x}} = (X \setminus A) \cup \{x\}$, then $S_{\mathbf{x}}$ is p-open, by assumption, a p-open cover $\{S_a : a \in \Lambda\}$ has finite sub cover, this shows that A is finite, and hence for any infinite subset of X has non empty interior. Conversely, let $\{S_a : a \in \Lambda\}$ be open cover of X, then $\{ \overline{S_\alpha}^o : a \in \Lambda\}$ is p-open cover of X, but X is compact, so that there is finite subcover, and by the property that for any infinite subset of X has non empty interior. We have that $\overline{S_\alpha}^o \setminus S_a$, is finite for each $a \in \Lambda$, hence there is a finite subset F of X, such that $X = \cup\{S_a : a \in \Lambda\} \cup F$, therefore (X, t) is p-compact.

4. Third main results

In this section, we will give the generalization of Tychonoff proved theorem, First we have presented the generalization of some of the basic notions for the product topology, finite intersection property,…and maximal superclass because we needed, so we say that a space X has finite intersection property, if every finite subclass $\{A_{i1}, A_{i2},..,A_{im}\}$ has non-empty intersection, and so on. Also we can define a semi-compact (p-compact, α-compact, β-compact) space X, as a space that it has the finite intersection property, for any class of semi-closed (p-closed, H-p-closed, α-closed, β-closed) subsets of X, with the finite intersection property. We have the following useful result.

Theorem 1.4.

Let $\{A_i : i \in I\}$ be a collection of semi-compact (p-compact, α-compact, β-compact) spaces, then the product space $X=\prod\{A_i : i\in I\}$, is also semi-compact (p-compact, α-compact, β-compact) space.

Proof

Let Д$=\{F_j\}$, be a class of semi-closed (p-closed, α-closed, β-closed) subsets of X, with the finite intersection property. Let $\mu=\{M_k ; k\in k\}$, be a maximal superclasses of Д with the finite intersection property. Define $\bar{\mu}=\{\overline{M}_k , k\in k\}$, since $F_j \in$ Д $\Rightarrow F_j=\overline{F}_j$ and $F_j\in \mu \Rightarrow F_j\in \bar{\mu}$. So if we prove that $\bar{\mu}$ has non-empty intersection, then Д will be has non-empty intersection. Now we will prove that, $\exists$ p∈X, s. t. $p\in \bar{\mu}_k$ $\forall k$. Since $\mu=\{M_k ; k\in k\}$, has the finite intersection property. Then for each projection $\pi_i : X\rightarrow A_i$, the class $\{\pi_i [M_k]: k\in k\}$, of subset of the coordinate space A_i, also has the finite intersection property, so the class of closures $\{\overline{\pi_i[M_k]}, k\in k\}$ is a class of closed subset of A_i, with the finite intersection property, but by hypothesis A_i, is semi-compact (p-compact, α-compact, β-compact). Then $\{\overline{\pi_i[M_k]}, k\in k\}$ has non-empty intersection. In other words there exist $a_i\in A_i$, s. t. $a_i\in \overline{\pi_i[M_k]}$ for each k, so if G_i, is any semi-open (p-open, α-open, β-open) subset of the coordinate space A_i, then there exist $a_i\in A_i$, s. t. $a_i\in G_i$, and $G_i \cap\pi_i [M_k]\neq\phi$, for every k, and hence if $p=\{a_i : i\in I\}$, then $p\in \overline{S}_\alpha^o$ $\forall k$, thus the proof is complete.

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